A Semi-Parametric Approach To Maintenance Optimization

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Abstract

We consider a deteriorating system subject to repair given partial information. The evolution of the system is described by a non-homogeneous Markov process. Between replacements the intensity of occurrence of minimal repairs is modulated by a measurable deterministic control function. The revenue depends on the system state and the costs depend on the level of control. Given the control value the revenue over a finite horizon is obtained.The expected revenue is maximized by choosing an optimal control strategy using a deterministic Hamilton-Jacobi equation.

1 INTRODUCTION

We optimize the revenue from a system subject to repair and maintenance. The system is subject to stochastic deterioration when working and can be in one of two unobservable states. The observed history of the process $\mathcal{F}_t^N$ is characterized by the counting process N_t of minimal repair points. Using the approach of Jensen (**?**), the intensity of minimal repairs is described by a stochastic measure which depends on the system state X_t. The flow of the process consists of a mixture of stochastic motions resulting from the natural motion of the process which are controlled by a non-homogeneous Markov process and the random jumps of the $\mathcal{F}_t^N$-adapted control process $u_t,\ t \in R_+$. To model the effect of covariate values, we consider the proportional intensity model (PIM) which is a generalized case of the proportional hazard model introduced by (**?**). By using the Filtering theorem (**?**) an explicit solution to the estimation of the system state with respect to F_t^N is derived. This solution transforms the control problem above into an intensity control model adapted to the observable history. By using a deterministic Hamilton-Jacobi Equation, in a Markovian manner, an optimal control process with respect to the counting process of minimal repairs is obtained.

2 MODELLING (PSP)

The evolution of the process and the effect of the environmental factors on the system are incorporated in the system state process (X_t), $t \in \mathbb{R}^+$ with state space $S = \{1, 2, ..., m\}$ whose transition between states is controlled by the non-homogeneous Markov process with time dependent transition probabilities and intensities matrix $P(t) = (p_{ij}(t))$ and $Q(t) = (q_{ij}(t))$. The occurrence of minimal repairs is described by the counting process $(N_t), t \in \mathcal{R}^+, N_t = \sum_{n\geq 1} I_{\{T_n \leq t\}}$ defined on a measurable space $(\Omega, \mathcal{F})$ where $T_1 < T_2 < ... < T_n < \ldots$ is the sequence of repair time points. It is assumed that the intensity of occurrence of repairs is a stochastic measure λ_{X_t} with smooth semi-martingale representation (SSM):

$$N_t = \int_0^t \lambda_{X_s} ds + M_t, \quad t \in \mathcal{R}_+, M \in \mathcal{M}_0 \tag{1}$$

Where $0 < \lambda_i < \infty, i \in S$, $\lambda_{X_t}, t \in \mathcal{R}^+$ is $\mathcal{F}$ progressive, $\mathcal{M}_0$ refers to the class of $\mathcal{F}$-martingales with paths which are right-continuous and left-limited (with $M_0 = 0$). The function $\varphi_t(i) = I_{\{X_t = i\}}$ is the indicator process of the system state. We assume that the minimal repair process is stopped by an $\mathcal{F}_t$-adapted failure time. If the lifetime of the system is measured by the $\mathcal{F}_t$-stopping time ξ, then $N = (N_t)$ is transformed to the process $N^\xi = (N_{t\wedge\xi})$ limited to the stopping time ξ. The process N_t^ξ again admits following $\mathcal{F}_t$-SSM representations:

$$N_t^\xi = \int_0^t I(\xi > s)\lambda_{X_s} ds + M_t^\xi = \int_0^t \sum_{i\in S} \lambda_i I(X_t = i, \xi > t) + M_t^\xi \tag{2}$$

where M_t^ξ is an $\mathcal{F}_t$-stopped martingale. The indicator process $Z_t = I(\xi \le t)$ describes the evolution of the system lifetime and $\lambda_t^\xi = \lambda_{X_t} I(\xi > t)$ can be realized as the $\mathcal{F}_t$-intensity of N_t randomized by the random failure time ξ. To model the lifetime of the process we assume that Z_t admits a smooth $\mathcal{F}$-semimartingale representation with an $\mathcal{F}$-martingale $M \in \mathcal{M}_0$ such that its regression part is the $\mathcal{F}_t$-adapted Cox process $\gamma_t \equiv \gamma(t, X_t)$ where $\gamma(t, X_t) = \lambda_0(t)\psi(X_t)$, $t \in R^+$ where both the baseline intensity $\lambda_0(t)$ and $\psi(x)$ are bounded and non-decreasing. In terms of the above measures the SSM representation of (Z_t) is given by

$$I(\xi \le t) = \int_0^t I(\xi > s)\gamma_s ds + M_t^\gamma, \quad t \in R^+ \tag{3}$$

where $(\gamma_t), t \in \mathcal{R}^+$ is a progressively measurable with respect to the filtration $\mathcal{F}_t$ with $E\int_0^t |\gamma_s| ds < \infty$ for all $t \in R^+$ and $M^\gamma = (M_t^\gamma) \in \mathcal{M}_0$.

We solve the intensity control with partial information using the projected version of the indicator process $(\varphi_t), t \in \mathcal{R}^+$ given the sub-filtration $\mathcal{F}_t^N = \sigma\{N_s : 0 < s \le t\}$, that is, $\hat{\varphi}_t(i) = E(\varphi_t(i)|\mathcal{F}_t^N)$ for $i \in S$. We use the smooth semi-martingale representation of $I(\xi \le t)$ with respect to the sub-filtration $\mathcal{F}^N$ to change the information level and the state we use the projection theorem (**?**). For $T_{N_t} \le t < T_{N_t+1}$ the projection formula yields the $\mathcal{F}_t^N$-SSM representation:

$$\bar{R}(N_t, t - T_{N_t}) = \int_0^t \sum_{i\in S} \gamma(s,i)\hat{\varphi}_s(i) R(N_s, s - T_{N_s}) ds + \bar{M}_t^\gamma, \tag{4}$$

where $R(N_t, t - T_{N_t}) = 1 - \bar{R}(N_t, t - T_{N_t})$ is the $\mathcal{F}_t^N$-adapted conditional survival probability, $R(N_t, t - T_{N_t}) = E\left(I(\xi \ge t)|\mathcal{F}_t^N\right)$, and $\hat{\gamma}_t(N) = \sum_{i\in S} \hat{\gamma}(t,i)\hat{\varphi}_t(i) R(N_t, t - T_{N_t})$ is $\mathcal{F}_t^N$-progressively measurable with $\hat{\gamma}_t(N) = E(I(\xi > t)\gamma_t|\mathcal{F}_t^N)$. Since $R(N_t, t - T_{N_t})$ has continuous paths of bounded variation the martingale term $\bar{M}_t^\gamma$ is identically zero, the solution of the reduced integral equation [4] is

$$R(N_t, t - T_{N_t}) = \exp\left(-\int_0^{(t-T_{N_t})} \sum_{i\in S} \gamma(s,i)\hat{\varphi}_s(i) ds\right) = \prod_{i\in S} R(N_t, i, s - T_{N_t}) \tag{5}$$

where $R(N_t, i, s - T_{N_t}) = \exp\left(-\int_0^{(t-T_{N_t})} \gamma(s,i)\hat{\varphi}_s(i) ds\right)$. We focus on modelling the inter-arrival times distribution of minimal repairs. We use an alternative definition of a doubly stochastic poisson process (**?**). The intensity term $\hat{\lambda}_t^\xi(N)$ in terms of $R(N_t, i, t - T_{N_t})$ can be written

$$\hat{\lambda}_t^\xi(N) = E(\lambda_t^\xi|\mathcal{F}_t^N) = \left(\sum_{i\in S} \lambda_i \hat{\varphi}_t^N(i)\right)\left(\prod_{i\in S} R(N_t, i, s - T_{N_t})\right) \tag{6}$$

Where $\hat{\varphi}_t^N(i) = \hat{\varphi}_t(i) I\{T_{N_t} \le t < T_{N_t+1}\}$. Now, for each $n \in \mathbb{N}_0$ let $\bar{F}_n(v)$ be the regular conditional distribution of the inter-arrival times $V_{n+1} = T_{(n+1)} - T_n$, $T_0 = 0$ given $\mathcal{F}_t^N$-adapted measure $\hat{\lambda}_t^\xi(n)$. Then,

$$\bar{F}_n(v) = p(V_{n+1} \ge v|\hat{\lambda}_t^\xi(n)) = \exp\left(-\int_{T_n}^{T_n+v} \sum_{i\in S} \lambda_i R(n, i, t - T_n)\hat{\varphi}_t^n(i) dt\right) \tag{7}$$

Where $T_n \le t < T_{n+1}$ for $n = 0, 1, 2, ...$

If η_{n+1} denotes the expected value of $(n+1)^{\text{th}}$ inter-arrival minimal repair time then by using the equation [7] we have

$$\eta_{n+1} = \int_0^\infty \bar{F}_n(v) dv = \int_0^\infty \exp\left(-\int_{T_n}^{T_n+v} \sum_{i\in S} \lambda_i R(n, i, t)\hat{\varphi}_t^n(i) dt\right) dv \tag{8}$$

Since the integrand term of equation (8)depends on T_n, η_{n+1} is a random measure. To deal with this we restrict ourselves to an estimated version of η_{n+1}. We use μ_{n+1} the $(n+1)^{\text{th}}$ expected value of repair times $\mu_{n+1} = E(T_{n+1}) = \sum_{k=1}^{n+1} \hat{\eta}_{n+1}$

$$\hat{\eta}_{n+1} = \int_0^\infty \hat{\bar{F}}_n(v)dv = \int_0^\infty \exp\left(-\int_{\mu_n}^{\mu_n+v} \sum_{i\in S} \lambda_i R(n,i,t)\hat{\varphi}_t^n(i)dt\right) dv \quad \text{for } n > 0\,. \tag{9}$$

We close this section by stating an explicit solution for the probability of the system state $\hat{\varphi}_t(i), i \in S$ adapted to the partial information $\mathcal{F}_t^N$. The filtering theorem gives an explicit solution

$$\hat{\varphi}_t(j) = \hat{\varphi}_0(j) + \int_0^t \left(\sum_{i\in S} \hat{\varphi}_s(i)\left\{q_{ij}(s) + \hat{\varphi}_s(j)(\lambda_i - \lambda_j)\right\}\right) ds + \sum_{n\geq 1}\left(-\hat{\varphi}_{T_n^-}(j) + \frac{\lambda_j \hat{\varphi}_{T_n^-}(j)}{\sum_{i=1}^m \lambda_i \hat{\varphi}_{T_n^-}(i)} I_{\{T_n \geq t\}}\right), \tag{10}$$

where $\hat{\varphi}_{t^-}(j)$ refers to the left limit.

3 MODELLING INTENSITY CONTROL

We assume that the intensity is controlled by a control function u and yields a revenue $\mu_{X_t}^u$ which depends on the system state and is subject to a cost $k_t(u)$ which depends on the control level. To set up the PSP model modulated by the control measure and to optimize the maintenance process, assume that $\mathcal{U}$ is the set of $\mathbb{R}^+$-valued measurable control processes of the form $u_t = u(t, N_t(\omega))$ where for each $n \in N^+$, $u(t,n)$ is $\mathcal{F}_t^N$-predictable and $u_t \in \mathcal{U}, t \geq 0, \omega \in \Omega$ To each control $u \in \mathcal{U}$ we associate a probability measure (*control dynamics*) $P_u, u \in \mathcal{U}$ on $(\Omega, \mathcal{F})$. It is assumed that N_t through transition rate $q_1(t)$ admits a $(P_u, \mathcal{F}_t^N)$-intensity $\lambda_t^\xi(u)$ of the form $\lambda_t^\xi(\omega, u) = \lambda(t, N_t(\omega), u_t(\omega))$, so that $q_1^u(t) = q_1(t,u)$. Thus, $q_1^u(t)$ can be regarded as a key tool to turn the maintenance template into the intensity control model. In addition, to each $u \in \mathcal{U}$ we associate a nonnegative measure $J(u)$:

$$J(u) = E_u\left[\int_0^T \left(\mu_{X_t}^u - k(t, N_t, u)\lambda_t^\xi(u)\right) dt - \phi_{X_T}^u\right] \tag{11}$$

T is a positive time, $\mu_{X_t}^u$ is a nonnegative $\mathcal{F}_t^N$-progressive process and $k_{T_n}(u)$, $\phi_{X_T}^u$ are nonnegative $\mathcal{F}_t^N$ predictable, and $\mathcal{F}_T$-measurable random variable respectively. The measure $J(u)$ associated to u is the value function, $\mu_{X_t}^u$, indexed by the system state is the reward per unit of time such that $(\mu_2 < \mu_1)$. This means raising the wear level of the system decreases the revenue obtained over inter-arrival time. The minimal repair cost at the n-th repair is $k_{T_n}(u)$ and cost of maintenance at the action time and $\phi_{X_T}^u$ is the final cost of replacement and inspection at terminal time T. By projection on the observed history of the process the $\mathcal{F}_t^N$-adapted version of $\hat{J}(u)$ is

$$\hat{J}(u) = E_u\left[\int_0^T \left(\sum_{i\in S} \mu_i \hat{\varphi}_t^u(i) - k(t, N_t, u)\sum_{i\in S} \lambda_i \hat{\varphi}_t^u(i) R(t, i, t - T_{N_t})dt - \sum_{i\in S} \phi_i^u \hat{\varphi}_T(i)\right)\right] \tag{12}$$

As seen, the intensity control problem is subject to *Markovian Controls*. More precisely, $J(u)(\forall u \in \mathcal{U})$ is characterized just with respect to $\mathcal{F}_t^N$-adapted measure $\hat{\varphi}_t$, or equivalently N_t. Following we restrict ourselves to solving the optimal control problem over finite horizon in a *Markovian Control* manner. To achieve this aim which is to select an optimum strategy $\{u_t^* \equiv u_t^*(t,n) : u_t^* \in \mathcal{U}\}$ so that $\hat{J}(u^*) = \sup_{u\in\mathcal{U}} \hat{J}(u)$ the "deterministic Hamilton-Jacobi Equation" (**?**) is employed.

Corollary 1. *Let for the measures $\hat{\lambda}^\xi(t,n,u)$, $\hat{\varphi}_t^u(i) = \hat{\varphi}(t,n,u)(i)$, and $k(t,n,u)$ where are independent of ω there exists for each $n \in N_+$ a function $V(t,n)$ such that*

$$\frac{\partial V(t,n)}{\partial t} + \sup_{u\in U_t}\left\{\hat{\lambda}^\xi(t,n,u)\left[V(t,n) - V(t,n-1) - k(t,n,u)\right] + \sum_{i\in S} \mu_i \hat{\varphi}(t,n,u)(i)\right\} = 0, \tag{13}$$

Given $V(T,n) = \sup_{u\in\mathcal{U}} \phi(T,n,u)$ where $\phi(T,n,u) = \sum_{i\in S} \phi_i \hat{\varphi}_T^u(i)$.

Suppose also that there exists for each $n \in N_+$ a measurable R_+-valued function $u^*(t,n)$ such that $u^*(t,n) \in \mathcal{U}, \quad t \in [0,T]$, and

$$u^*(t,n) = \operatorname*{argmax}_{u \in U_t} \left\{ \hat{\lambda}^{\xi}(t,n,u) \left[V(t,n) - V(t,n-1) - k(t,n,u) \right] + \sum_{i \in S} \mu_i \hat{\varphi}(t,n,u)(i) \right\}, \qquad (14)$$

Then u_t^* defined by $u_t^*(\omega) = u^*(t, N_t(\omega))$ for $\omega \in F_t^N$ is an optimal solution.

4 NUMERICAL RESULTS

To obtain an optimal control solution for the deteriorating model presented above, a simple linear transition rate is assumed, $q_1(t) = t$ and the repair cost is a function of the state probability and control action, $k(t,n,u) = K - C\hat{\varphi}(t,n,u)(1)$ $(0 < C < K)$. It means with raising the impairment level of the system, the repair cost decreases. The failure trend of is assumed to be linear $\psi(x) = x$ and the baseline function is Weibull with intensity function $\lambda_0(t) = \dfrac{\alpha t^{\alpha-1}}{\beta^\alpha}, \quad t \geq 0$. The ordinary differential equation [13] is solved using the Euler method with step size $h = 0.1$ together with equation [9]. The optimal control process and the optimal control intensity with parameters $\mu_1 = 2$, $\mu_2 = 1$, $\lambda_2 = 2$, $\lambda_1 = 1$, $k = 2$, $c = 1$, $\phi_1 = 1$, $\phi_2 = 2$, $\alpha = 2$ and $\beta = \sqrt{2}$ are shown in Figures [1] and [2]).

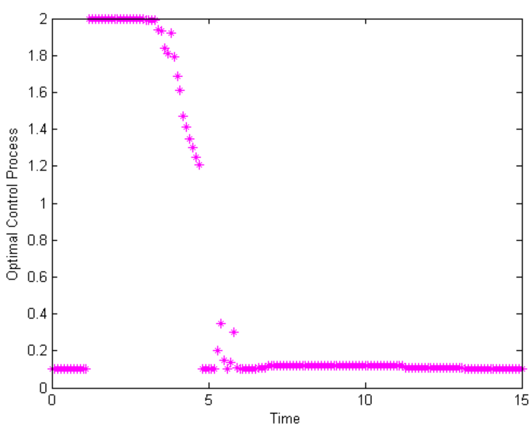

Figure 1: An evolution of the optimal control process u_t^* over [0,15]

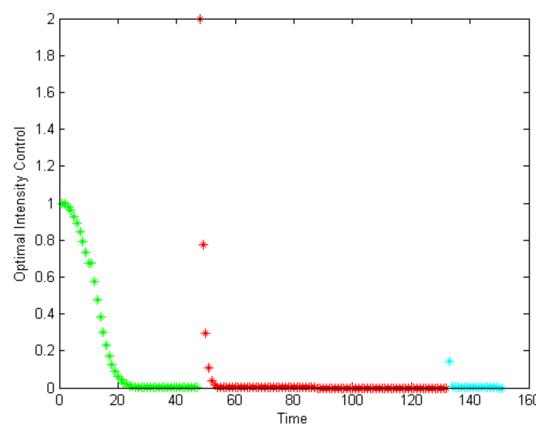

Figure 2: An evolution of the optimal intensity control given u_t^*

References