Multi-state Reliability Systems under Discrete Time Semi-Markovian Hypothesis

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Abstract

We consider repairable reliability systems with m components, the lifetimes and repair times of which are independent. The l-th component can be either in the failure state 0 or in the perfect state d_l or in one of the degradation states $\{1,\ 2,\ \ldots,\ d_l - 1\}$. The time of staying in any of these states is a random variable following a discrete distribution not geometric. Thus, the state of every component and consequently of the whole system is described by a discrete-time semi-Markov chain together with the backward recurrence chain. Using recently obtained results concerning the discrete-time semi-Markov chains, we derive basic reliability measures in a general form. Finally, we present some numerical results of our proposed approach in specific reliability systems.

1 The General Model

Let us consider a multi-state system (MSS) of order m, which means that it consists of m components which in their run are multi-state. As usually, the states of components determine the state of the system. We assume that the system state space and the one of the component l, $1 \leq l \leq m$ are defined by

$$E = \{0,\ 1,\ \ldots,\ d\} \text{ and } E_l = \{0,\ 1,\ \ldots,\ d_l\},$$

where 0 means complete failure of the system and of the component l and the states d and d_l denote a perfect functioning of the system and of the component l respectively, while the between states are degradation states. For a general reference see Lisnianski and Levitin (2003), as well as the references therein.

Our aim is to study basic reliability measures under the assumption that the evolution in time of the above MSS is governed by a discrete time semi-Markov chain.

We proceed to our analysis by examining first the behavior of a component. Thus, let l be a component of a system with state space E_l, $1 \leq l \leq m$. As we mentioned earlier, the state d_l is the perfect state, 0 is the failure state, while the others are degradation states. We associate to this component (l) a discrete time semi-Markov chain $\{Z_k^l, k \geq 0\}$ with state space E_l, which describes the evolution in time of the component. Considering the associated homogeneous Markov Renewal Chain $\{(J_n^l,\ S_n^l), n \geq 0\}$, we denote the corresponding semi-Markov kernel by

$$q_{ij}^l(k) = \mathbb{P}(J_{n+1}^l = j, S_{n+1}^l - S_n^l = k | J_n^l = i), \quad i,\ j \in E_l,\ k \in \mathbb{N},\ n \in \mathbb{N},$$

where $\{J_n^l\}_{n\in\mathbb{N}}$ is the embedded Markov chain and $\{S_n^l\}_{n\in\mathbb{N}}$ the jumps times. We assume that $S_0 = 0$ and $0 < S_1 < \ldots < S_n < S_{n+1} < \ldots$ (see Barbu and Limnios (2008)). A discrete time semi-Markov chain $\{Z_k, k \geq 0\}$ has been studied by O. Chryssaphinou, M. Karaliopoulou and N. Limnios (2008). They defined the backward recurrence time for the chain $\{Z_k, k \geq 0\}$, which is as follows in our case:

Definition 1. For any $k \in \mathbb{N}$, the backward recurrence time $\{U_k^l\ k \geq 0\}$ of the semi-Markov chain $\{Z_k^l;\ k \geq 0\}$ is defined by

$$U_k^l = k - S_{N(k)},$$

where $N(k) = \max\{n : S_n \leq k\}$.

The following example clarifies the above definition. For $d_l = 3$, let us assume the following sequence of states for component l (starting from the perfect state) 3 3 3 3 2 2 1 1 0 ... . In this case, the backward recurrence times are $U_0 = 0,\ U_1 = 1,\ U_2 = 2,\ U_3 = 3,\ U_4 = 0,\ U_5 = 1,\ U_6 = 0,\ U_7 = 1,\ U_8 = 0,\ \dots$, while $(J_0 = 3,\ S_0 = 0),\ (J_1 = 2,\ S_1 = 4),\ \dots$.

The stochastic process $\{(Z^l_k, U^l_k);\ k \geq 0\}$, $l = 1,\ \dots\ m$ with state space $\Theta^l = E_l \times \mathbb{N}$ is a discrete time Markov chain with transition probabilities of the first order given by the following theorem.

Theorem 1. *The first order transition probabilities of the Markov chain* $\{(Z^l_k, U^l_k);\ k \geq 0\}$ *are*

$$\mathbb{P}^l(Z^l_{k+1} = j,\ U^l_{k+1} = u' \mid Z^l_k = i,\ U^l_k = u) =$$
$$= \begin{cases} \dfrac{q_{ij}(u+1)}{1 - H_i(u)}, & \text{if} \quad u' = 0, \\ \dfrac{I_{\{i=j\}}[1 - H_i(u+1)]}{1 - H_i(u)}, & \text{if} \quad u' = u+1, \\ 0, & \text{elsewhere,} \end{cases}$$

for every $(i, u),\ (j, u') \in \Theta^l$ *and for every* $k \in \mathbb{N}$ *such that* $H_i(u) < 1$.

(see Chryssaphinou, M. Karaliopoulou and N. Limnios (2008))

We note that if the sojourn times in different states are of finite support, then the corresponding transition probability matrices $\tilde{P}^l$, $l = 1,\ 2\ \dots\ m$ are finite, otherwise they are infinite.

We now return to the system with m components as it is described above. We have that the stochastic process

$$\xi \equiv \{\tilde{\xi}_k; k \geq 0\} = \{((Z^1_k, U^1_k);\ \dots\ ; (Z^m_k, U^m_k)), k \geq 0\}$$

is a discrete Markov chain with transition probability matrix

$$\tilde{P} = \tilde{P}^1 \otimes \quad \cdots \quad \otimes \tilde{P}^m,$$

where $\otimes$ denotes the Knonecker product of matrices. We recall that for two matrices $A = (a_{ij})$ and B it holds that

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B & \dots \\ a_{21}B & \dots & a_{2n}B & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The discrete time semi-Markov chain ξ has state space $\prod_{l=1}^m (E_l \times \mathbb{N})$, which is enumerated into a lexicographic order.

Now, we consider the transformation, which maps the space of possible combinations of states of all components into the state space of the system

$$\phi : \prod_{l=1}^{m} E_l \to E.$$

Clearly, this is the structure function of the system. It is

$$(x_1,\ \dots\ , x_m) \to \phi(x_1,\ \dots\ , x_m),\ x_l \in E_l, \quad l = 1,\ \dots, m.$$

Defining a value $r \in E$, $0 < r \leq d$, which can be called r-level of the system, we say that the system is reliable at time k if it operates at least at level r. So, in terms of structure function, it means that we deal with the following subspace of possible combinations of states of all components

$$L_r := \{(x_1,\ \dots,\ x_m) : \phi(x_1,\ \dots,\ x_m) \geq r\}.$$

Let us set T_r to be the time to failure at level r as

$$T_r = \inf\{k : (Z^1_k,\ \dots,\ Z^m_k) \in \overline{L}_r\},$$

where $\overline{L}_r$ denotes the complement of L_r. Clearly, the sets

$$E^U = \{((x_1, u_1), \ \cdots (x_m, u_m)) : (x_1, \ \ldots, \ x_m) \in L_r, \quad u_l \in \mathbb{N}\},$$

$$E^D = \{((x_1, u_1), \ \cdots (x_m, u_m)) : (x_1, \ \ldots, \ x_m) \in \overline{L}_r, \quad u_l \in \mathbb{N}\}$$

consist the sets of up and down states respectively and they are such that $E = E^U \cup E^D$, $E^U \cap E^D = \emptyset$, $E^U \neq \emptyset$, $E^D \neq \emptyset$. We denote by $\tilde{P}_{11}$ and $\tilde{P}_{22}$ the restriction of the matrix $\tilde{P}$ on $E^U \times E^U$ and on $E^D \times E^D$ respectively. Finally, considering the initial probability vector $\tilde{\alpha}$ we get the analogous restrictions $\tilde{\alpha_1}$ and $\tilde{\alpha_2}$.

Theorem 2. *Under the above assumptions and notations the reliability, availability and maintainability of the MSS system are given by the following relations respectively*

$$R_r(k) = \tilde{\alpha_1} \ \tilde{P}_{11}^k \ \tilde{1}, \quad A_r(k) = \tilde{\alpha} \ \tilde{P}^k \ \tilde{1}_{1,0} \quad \textit{and} \quad M_r(k) = 1 - \tilde{\alpha}_2 \ \tilde{P}_{22}^k \ \tilde{1}.$$

By $\tilde{1}_{1,0}$ we denote a vector whose elements are equal to 1 for the up states and the rest are equal to 0.

Moreover, the mean time to failure (MTTF) and the mean time to repair (MTTR) are given by the following relations respectively

$$MTTF = \tilde{\alpha_1} \ (\tilde{I} - \tilde{P}_{11})^{-1} \ \tilde{1} \quad \textit{and} \quad MTTR = \tilde{\alpha_2} \ (\tilde{I} - \tilde{P}_{22})^{-1} \ \tilde{1},$$

under the assumption of considering finite support for the sojourn time distribution and that the matrices $\tilde{I} - \tilde{P}_{11}$, $\tilde{I} - \tilde{P}_{22}$ are not singular.

The proofs of the above relations can be found in Barbu and Limnios (2008).

We note that other basic measures as failure rate, mean up time, mean down time, etc. can be explicitly derived. To simplify the complexity of the computation of the above measures, we can consider truncated sojourn times distributions, i.e. for a sojourn time distribution $f(k), \ k \in \mathbb{N}$ we can get $f(k), \ 0 \leq k \leq M$, such that $\sum_{k>M} f(k) < \varepsilon$, for a predefined small $\varepsilon > 0$. Thus, the resulting error concerning the reliability of the system could be of order $O(\varepsilon)$.

2 Applications

In this section we will present the main reliability measures of four different repairable systems under the discrete time semi-Markov hypothesis. We assume that each system consists of three components, which function and are repaired independently of each other. Moreover, we assume that the current condition of each component of the system (up or down) is described by a discrete time semi Markov chain with state space $E = \{0, 1\}$ (the down state is denoted by 0 and the up state by 1) and with semi Markov kernel

$$\mathbf{q}_{01} = \{0.1, \ 0.2, \ 0.4, \ 0.2, \ 0.1\} \text{ and } \mathbf{q}_{10} = \{0.05, \ 0.05, \ 0.10, \ 0.50, \ 0.30\}.$$

Table 1: Mean time to failure (MTTF) for the proposed systems

Systems	MTTF
serial	3.1254
2-out-of-3:F	4.1677
consecutive 2-out-of-3:F	4.6552
parallel	10.2981

In this case, the resulting Markov process has as state space the set $E = (\{0, 1\} \times \{0, 1, 2, 3, 4\})^3$. Thus, the dimension of the corresponding transition matrix is 1000×1000. According to the structure of the system under consideration, the state space is partitioned into two subsets that concern the up and the down states. In our examples there are 125, 500, 625 and 875 up states for the series, the 2-out-of-3:F, the consecutive-2-out-of-3:F and the parallel system respectively. According to the partition of the state space into up and down states, the transition matrix $\tilde{P}$ and the initial distribution $\tilde{\alpha}$ are also partitioned. Based on the above deductions, we calculate the reliability, the availability and the mean time to failure for each of these systems.

In Figure 1, the reliability $R(n)$ of the four systems is depicted. As it was expected, the series system has the lowest reliability for all k, while the parallel system that has the highest one. The availability of the four systems is presented in Figure 2. Finally, the mean times to failure for the studied systems are given in Table 1.

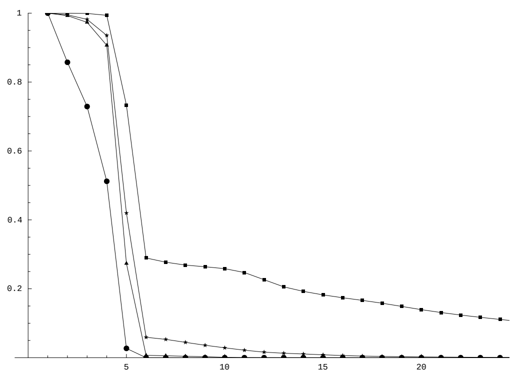
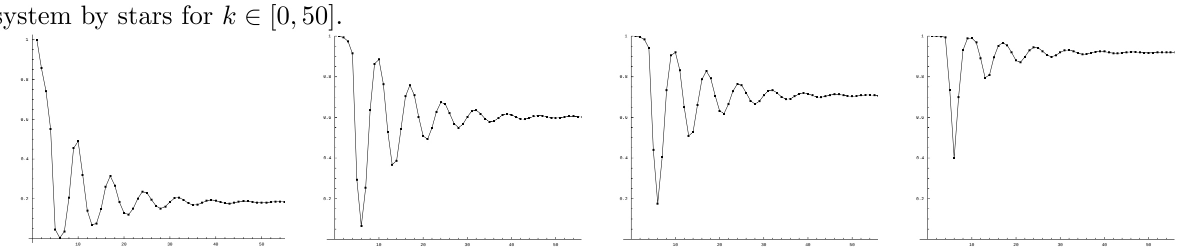


Figure 1: Reliability $R(k)$ of the four systems. The reliability of the series system is depicted by circles, of the parallel system by boxes, of the 2-out-of-3:F system by triangles and of the consecutive-2-out-of-3:F system by stars for $k \in [0, 50]$.



(a) (b) (c) (d)

Figure 2: Availability of the (a) series system, (b) 2-out-of-3:F system, (c) consecutive-2-out-of-3:F system and (d) parallel system.

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