Recursive Equations of Reliability for Linear Consecutive-k-out-of-n:F Systems with Sparse d

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Abstract

In this paper, some recursive equations of reliability for linear consecutive-k-out-of-n: F systems with spares d are introduced. The system of consecutive failures with sparse d is a generalized extension of consecutive-k systems, which was proposed by Zhao *et al.* (2007) in which the reliability formulas of these systems were presented by using finite Markov Chain imbedding approach. To facilitate the use, two theorems and some special cases on recursive equations for the system reliability are given.

1. Introduction.

The system reliability model is a basis of reliability analyses, so that it plays an important role in both theory and practice. Since 1980's, the consecutive-k systems have been widely studied, which can be used to model the microwave station network, oil pipeline systems, vacuum systems in an electron accelerator, and photography of a nuclear accelerator. There is a lot literature on it, for example, such as Koutras (1996), Chang *et al.* (1999, 2000), Cui *et al.* (2006), and others. Zhao *et al.*(2007) introduced a generalized consecutive-*k* system, and named them as consecutive-k systems with sparse d. The consecutive failures with sparse d is defined as follows. When components are ordered in a line, two adjacent failed components have no failed components between them, and the number of working components between the two failed components is d, then the two failed components are called consecutive failures with sparse d. For a linear reliability system, if there are $j\ (j \geq 2)$ failed components, then there are $j-1$ adjacent failed component pairs and $j-1$ nonnegative integers $d_i, (i=1,2,\ldots, j-1)$ which are the numbers of working components between two adjacent failed components, and we call the system has consecutive j failures with sparse $d = \max\{d_i, i=1,2,\ldots, j-1\}$. A consecutive-$k$-out-$n:F$ system with sparse d consists of n components ordered in a line, and the system fails iff, there exist at least k consecutive failures with sparse d. It is clear that a consecutive-k-out-$n:F$ system with sparse d becomes into a common consecutive-k-out-$n:F$ system when $d=0$. Zhao *et al.* (2007) presented some system reliability formulas with matrix forms by using the finite Markov chain imbedding approach. However, the matrix form formulas have some disadvantages such as unclear relation on component reliability with the system reliability, less intuitive and large dimension when the system has more components. On the other hand, the recursive equations have been used more in computing system reliability, especially for the iid situations, because it is easy to be used to analyze system reliability. In this paper, two theorems and some special cases on recursive equations for the system reliability are given.

2. Recursive Equations.

First it is assumed that the independence condition is hold for all components' working and failures, and each component has two states: working and failure, the component reliability are $p_i\,(i=1,2,\dots,n)$. For a linear consecutive-k-out-of-n:F system $(k \ge 2)$ with sparse d, the following results are given.

Theorem 1. For a linear consecutive-k-out-of-n:F system with sparse d with n components independently each other, each component has two possible states: Working and Failure, then its system reliability satisfies the following recursive equation.

$$R_d(n,k;p_1,p_2,\dots,p_n) = R_d(n-1,k;p_1,p_2,\dots,p_{n-1}) - q_n \sum_{j=k+d+1}^{kd+k+1} \Delta_j R_d(n-j,k;p_1,p_2,\dots,p_{n-j}),$$

where $\Delta_j = P\{$ There are $k-1$ failed components among components $n-1, n-2, \dots, n-j+1$, and components $n-j+1, n-j+2, \dots, n-j+d+1$ are the first consecutive $d+1$ working component subsection accounted from right to left sides, component $n-j+d+2$ is failed$\}$, $R_d(n,k;p_1,p_2,\dots,p_n)$ is the system reliability.

Proof. Let the symbol $L_d(n,k;p_1,p_2,\dots,p_n)$ denote a linear consecutive-k-out-of-n:F system with sparse d with n components. Since if $L_d(n,k;p_1,p_2,\dots,p_n)$ works, then $L_d(n-1,k;p_1,p_2,\dots,p_{n-1})$ works too. On the other hand, $L_d(n-1,k;p_1,p_2,\dots,p_{n-1})$ works, then $L_d(n,k;p_1,p_2,\dots,p_n)$ works except for the cases which are described as follows.

Component n fails, and there are $k-1$ failed components among components $n-1, n-2, \dots, n-j+1$, and components $n-j+1, n-j+2, \dots, n-j+d+1$ are working which are the first consecutive $d+1$ working component section accounted from right to left sides, and component $n-j+d+2$ fails, where $j = k+d+1, \dots, kd+k+1$. Thus the theorem is proved.

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Special case for $d=0$, then we get

$$R_{d=0}(n,k;p_1,p_2,\dots,p_n) = R_{d=0}(n-1,k;p_1,p_2,\dots,p_{n-1}) - q_n q_{n-1} \cdots q_{n-k+1} p_{n-k} R_d(n-k-1,k;p_1,p_2,\dots,p_{n-k-1}),$$

which coincides with Corollary 2.1.2 at page 7 in Chang *et al*. (2000).

In order to understand theorem 2, here an example is discussed. For the case of $d=1, k=3$, we get,

$$R_1(n,3) = R_1(n-1,3) - q\sum_{j=5}^{7} \Delta_j R_1(n-j,3),$$

After simple calculations, we get $\Delta_5 = p^2q^2$, $\Delta_6 = 2p^3q^2$, $\Delta_7 = p^4q^2$, thus the system reliability is

$$R_1(n,3) = R_1(n-1,3) - p^2q^2R_1(n-5,3) - 2p^3q^2R_1(n-6,3) - p^4q^2R_1(n-7,3).$$

For the independent identically distributed (iid) cases, $(k \ge 2)$,

$$R_d(n,k) = R_d(n-1,k) - q\sum_{j=k+d+1}^{kd+k+1} \Delta_j R_d(n-j,k),$$

where $\Delta_j = a_j p^{j-k} q^{k-1}$, the combination coefficient a_j can be obtained in the following way, i.e. a_j is the numbers of the distinct solutions for the following equation,

$$\begin{cases} x_1 + x_2 + \cdots + x_{k-1} = j - d - k - 1, \\ 0 \le x_i \le d,\ x_i \in \mathbf{Z},\ i = 1, 2, \ldots, k-1. \end{cases}$$

Since components $n - j + d + 2$ and n are failed ones located at two ends in the section of components $n - j + d + 2, n - j + d + 3, \ldots, n$, there are $(j - d - 3) - (k - 2) = j - d - k - 1$ working components and there are not any consecutive $d + 1$ or more working component subsections among the component section. Thus

$$R_d(n,k) = R_d(n-1,k) - q^k \sum_{j=k+d+1}^{kd+k+1} a_j p^{j-k} R_d(n-j,k).$$

Special cases:

(i) $d = 1, k = 3,$ then $j = 5, 6, 7$,

For a_5, we can consider the equation $\begin{cases} x_1 + x_2 = 0, \\ 0 \le x_i \le 1,\ x_i \in \mathbf{Z},\ i = 1, 2. \end{cases}$, thus there is one solution for it, i.e., $x_1 = x_2 = 0$, that is, $a_5 = 1$. Similarly, we get $a_6 = 2, a_7 = 1$, thus we get

$$\begin{aligned} R_1(n,3) &= R_1(n-1,3) - q^3 \sum_{j=5}^{7} a_j p^{j-k} R_1(n-j,3) \\ &= R_1(n-1,3) - q^3 [p^2 R_1(n-5,3) + 2p^3 R_1(n-6,3) + p^4 R_1(n-7,3)]. \end{aligned}$$

Theorem 2. For a linear consecutive-k-out-of-n: F system with sparse d with n components independently each other, each component has two possible states: Working and Failure., then its system reliability satisfies the following recursive equation.

$$\begin{aligned} R_d(n,k; p_1, p_2, \ldots, p_n) &= p_n p_{n-1} \cdots p_{n-d} R_d(n-d-1, k; p_1, p_2, \ldots, p_{n-d-1}) \\ &+ \sum_{j=d+2}^{(d+1)k} \nabla_j R_d(n-j, k; p_1, p_2, \ldots, p_{n-j}), \end{aligned}$$

where $\nabla_j = P\{$ There are at most $k-1$ failed components among components $n, n-1, \ldots, n-j+1$, and components $n-j+1, n-j+2, \ldots, n-j+d+1$ are the first consecutive $d+1$ working component subsection accounted from right to left sides, component $n-j+d+2$ is failed$\}$.

Proof. In order to get the recursive reliability equation, we decompose the system working cases with the first consecutive $d+1$ working component section accounted from right to left sides. Thus the first case is that components $n, n-1, \ldots, n-d$ work, similarly, in general, components $n-j+1, n-j+2, \ldots, n-j+d+1$ are the first consecutive $d+1$ working component subsection accounted from right to left sides, component $n-j+d+2$ is failed, where $j = d+2, \ldots, (d+1)k$, and all cases are exclusive. Thus the theorem is proved.

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Special case for $d = 0$, then we get

$$\begin{aligned} R_{d=0}(n,k; p_1, p_2, \ldots, p_n) &= p_n R_{d=0}(n-1, k; p_1, p_2, \ldots, p_{n-1}) \\ &+ \sum_{j=2}^{k} p_{n-j+1} \prod_{i=n-j+2}^{n} q_i R_{d=0}(n-j, k; p_1, p_2, \ldots, p_{n-j}) \end{aligned}$$

which coincides with Corollary 2.1.4 at page 8 in Chang *et al*. (2000).

In order to understand theorem 2, here an example is discussed. For the case of $d=1, k=3,$ we get,

$$\begin{aligned}R_1(n,3;p_1,\ldots,p_n) = {} & p_n p_{n-1} R_1(n-2,3;p_1,\ldots,p_{n-2}) \\ & + p_{n-1}p_{n-2}q_n R_1(n-3,3;p_1,\ldots,p_{n-3}) \\ & + p_{n-2}p_{n-3}q_{n-1}R_1(n-4,3;p_1,\ldots,p_{n-4}) \\ & + p_{n-3}p_{n-4}[q_{n-2}q_{n-1}p_n + q_{n-2}p_{n-1}q_n]R_1(n-5,3;p_1,\ldots,p_{n-5}) \\ & + p_{n-4}p_{n-5}q_{n-3}p_{n-2}q_{n-1}p_n R_1(n-6,3;p_1,\ldots,p_{n-6}).\end{aligned}$$

when $p_i \equiv p,\ i=1,2,\ldots,n$. The special case above can be become into,

$$\begin{aligned}R_1(n,3;p) = {} & p^2R_1(n-2,3) + p^2qR_1(n-3,3) + p^2qR_1(n-4,3) + 2p^3q^2R_1(n-5,3) \\ & + p^4q^2R_1(n-6,3).\end{aligned}$$

3. Conclusions

Two recursive equations of reliability for linear consecutive-k-out-of n:F systems with spares d are given in this paper, and some special cases are discussed too, specially for the iid case, the recursive equation can be given by a polynomial of component reliability p. Two theorems presented in the paper can cover the results in the linear consecutive-k-out-of-n:F systems. Some results are also needed to be given, but the limitation of the length of conference paper can not be allowed to do this.

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