Warranty and Fair Pricing for Used Items in Two-Dimensional Life Time

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Abstract

Cost assessment related to age of multiple components is discussed. We use definitions of equivalent ages in multi-dimensional life, to illustrate how cost characteristics should be fairly assessed. Pricing of used items with residual value, and assigning warranty for used items as a function of item's age are the subjects of our study. Numeric examples based on correlated Bivariate normal distribution for two life components illustrate our concepts.

1 Introduction

Warranty studies are based on mathematical models of reliability and operating maintenance processes supposedly build close to assumed warranty policies. Background and discussions of various aspects of warranty policies and related issues can be found in Blischke and Murthy (1992, 1996). Any contemporary article on warranty usually contains a good motivation and a survey of recent publications (e.g. Chen and Popova 2008). An excellent summary of warranty economic decision models can be read in Thomas and Rao, 1999.

The multi dimensional warranty is relatively rare considered, although it is also one of the most offered in automotive business. Warranty is offered for cars at the purchase and is valid during some limited future time of use, or until some future mileage is driven, whichever comes first. The client can purchase such warranty even though the purchased car is used. Active studies on models for two-dimensional warranty have been actively studied by Mitra and Patankar (2001-2007), Murthy et al. 1995, and others. Models of higher dimension than 2 (to best of our knowledge), are not discussed in articles on warranty.

Our talk is focused on cases of two life components. We use an interpretation of age in multi dimensional life (Dimitrov 2007). It allows applications in pricing, and in assigning warranty. The approach can be used in higher dimensional age. It allows quick estimate of loses in the most conservative warranty expenses associated with used items. Examples with Bivariate Normal distribution illustrate the work of this approach.

2 Multi component life

The "life time" of almost any system might be measured in more than one time scale. Life is presented not just in terms of the system' calendar time of existence (since it has been put in operation) but also with certain additional measurements such as amount of work performed, internal resources wasted, growth, accumulated depreciation, energy exhausted, weariness, and other similar indicators. For the age assessment of airplanes there are at least three "life components" of interest, the calendar age, the amount of time flown (in the air) and the number of takeoffs and landings. These life components usually represent various resource variables related to the system and are positively dependent. At the time of death every one of these life variables $X, Y, \ldots$ have some measured values. The death itself is an event indirectly related to the sense of $X, Y, \ldots$ i.e. not all of these variables may have a direct ``time meaning''. Usually, the calendar time at death is counted as age of the item.

However, age is a relative concept and needs to be specifically treated. We use this concept (Dimitrov 2007), and discuss it in multi dimensional life. We confine our attention to systems with two life components (*X*, *Y*). The probability *P*($X \le x$, $Y \le y$) is their joint cumulative distribution function (c.d.f.) $F_{X,Y}(x, y)$. Its value indicates the proportion of individual systems from the population which will have $X \le x$ and $Y \le y$ when the system dies.

The probability to survive both values (*x, y*), defines the *Survival Function* $S_{X,Y}(x, y)$ of the system. It indicates the proportion of individual systems from the population which will have both values $X > x$ and $Y > y$ when die. The variables *X* and *Y* may be treated as two dimensional age, or can be just *collaterals to the age*. Their measurements give important information for users close to the ways in which calendar time is used in the risk assessment, and utility.

Definition 1 (Optimistic): Individuals 1 and 2 with two dimensional life components $(X_1, Y_1) = (T_1^{(1)}, T_2^{(1)})$ and $(X_2, Y_2) = (T_1^{(2)}, T_2^{(2)})$ have equivalent ages $(T_1^{(1)}, T_2^{(1)}) = (T_1^{(2)}, T_2^{(2)})$ if

$$F_{X_1,Y_1}(T_1^{(1)}, T_2^{(1)}) = F_{X_2,Y_2}(T_1^{(2)}, T_2^{(2)}).$$

Notice, that we now have sets *of equivalently aged individuals* from any two populations. Moreover, within one population the coordinates of the points on the curves

$$C_p := \{(x_p, y_p); F_{X,Y}(x_p, y_p) = p, p \in (0,1)\} \tag{1}$$

are tracing individuals of the same age. The sets C_p represent the *curves of equivalent ages* for the individuals in this population at level *p*. Individuals whose life components *X* and *Y* have measurements located on lower level curves are younger than individuals with measurements on higher level curves.

Definition 2 (Pessimistic): Individuals 1 and 2 with two component lives $(X_1, Y_1) = (T_1^{(1)}, T_2^{(1)})$ and $(X_2, Y_2) = (T_1^{(2)}, T_2^{(2)})$ have equivalent ages $(T_1^{(1)}, T_2^{(1)}) = (T_1^{(2)}, T_2^{(2)})$ if

$$S_{X_1,Y_1}(T_1^{(1)}, T_2^{(1)}) = S_{X_2,Y_2}(T_1^{(2)}, T_2^{(2)}).$$

We have sets of equivalent ages again, even in the frames of one population. The points on the curve

$$G_p := \{(x_p, y_p);\ S_{X,Y}(x_p, y_p) = p, p \in (0,1)\} \tag{2}$$

trace the individuals of equivalent ages from this population at survival level *p.* The individuals whose measured life components are on higher survival level curves are younger than individual with life component measurements on lower level curves of the survival function.

Example 1 We illustrate our considerations on a population with two component life with correlated Bivariate Normal distribution

$$f(x, y) = (2\pi\sigma_1\sigma_2\sqrt{1-\rho^2})^{-1} \exp\{-\frac{1}{2(1-\rho^2)}[(\frac{x-\mu_1}{\sigma_1})^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + (\frac{y-\mu_2}{\sigma_2})^2]\}$$

for $-\infty<x<\infty$, and $-\infty<y<\infty$, where μ_1, μ_2 arbitrary, $\sigma_1 > 0$, $\sigma_2 > 0$, and $-1<\rho<1$ are the five parameters of this distribution (expected values, standard deviations and correlation coefficient between the two random components *X* and *Y).* By altering these parameters one can model lots of practical dependence between two life components.

3 Fair Pricing of Used Items

Fair pricing of used items with residual value should follow the rules of the rebate warranty. If the original price of an item is K_0 and it is sold used with values on the curve of equivalent ages C_p, then it fair price needs not to exceed $K_0(1-p)$ - the pessimistic value. Therefore all items with life components laying on one and the same curve of equivalent ages must be the same. Discount factors proportional to the time elapsed since the first purchases are also possible. There are possibilities to make adjustments by giving weights on the particular components (x_0, y_0) accumulated by the used item during its history of usage. A combination between the two definitions of the age on the curve C_p where $p = F(x_0, \; y_0)$, or on the curve $q = S(x_0, \; y_0)$ also shows that if the used item happens on the age curve G_{1-q}, its price should be no less than $K_0(1-q)$ - the "optimistic value". Salvation value potentially makes the portion 1-*q* from the value of a new item. The two values of the functions 1-$F(x_0, \; y_0)$ and $S(x_0, \; y_0)$ serve as estimators of the portion of average resource left in an individual with measured accumulated levels (x_0, y_0) of its life components. Therefore, the fair pricing should be made by numbers between these two values. We illustrate this on an example with used cars.

Example 2 Assume *X*=57 (thousand miles) is the mileage of a car, and *Y*=3.5 year is its calendar age. We assume the age (*X,Y*) has the distribution of Example 1 with parameters μ_1= 220 thousands miles, μ_2= 12 years, σ_1= 65 thousand miles, σ_2=4 years, and correlation between the two measurements ρ = .75. The calculations based on Bivariate normal distribution with the above parameters show that this vehicle stays the curve $C_{.003009}$, and on the survivors curve $G_{.98014}$. The vehicles staying on the level curves $C_{.384973}$ and $G_{.384973}$. A new vehicle priced $20,000 at age (220,12) should be sold for a price between $7,699 and $12,300. A used item of age (57,3.5) and price as new $20,000, should have fair price (without discounts) between $19,602 and $19,939.

Kelly's Blue Book gives free web based advises about the fair prices of new and used cars at http://www.kbb.com/kbb/UsedCars/default.aspx for almost any brand of cars. After answering a series of questions one can get an advice what value should have the care of his/her personal interest. The most of the question are about extras, the car condition (excellent, good, fair) region where you live, and an ultimate price depending on the answers is given. Our discussion here insists that fair prices are advising values, and can be used in various fields where used items are subject of bargain, when age admits multi component interpretation.

4 Warranty for used items

Multi dimensional warranty policies allow more options than the one-variable case. Under these policies the customer is refunded portion of the sales price or the losses incurred because the items failure. The warranty is offered in the forms of "boxed values" guaranteed on each age component to be performed by the item (usually, by the phrase "which ever is fulfilled first"). The policy of Chen and Popova, 2008 combines most models. There are two pairs of numbers (V_X, V_Y) and (W_X, W_Y), where $0 \le V_X \le W_X$, and $0 \le V_Y \le W_Y$. If the system fails with age (*X,Y*)Є[0, V_X]x[0, V_Y] (denote this set by **A**), then it is replaced by a new one. Denote by **B=**[0, W_X]x[0, W_Y] the set of continued warranty. It contains set **A.** If the system fails with age belonging to the set **B** but outside **A**, then just a minimal repair will be funded. Difficulties in multi dimensional case come because authors are trying to use models for failure rates for the system. And it is known that failure rates are not uniquely defined, thus too many compromises are admitted.

We advise the use of joint distribution – the c.d.f.'s or the survival functions, and the concept of equivalent life. It is appropriate for warranty of non-repairable items, but may serve as advising tool in case of repairable items and costs depending of age components measured at the times of failure. Moreover, our approach works also in case of used items.

Assume, the age of the system at the purchase is given by the pair ($X = x_0, Y = y_0$), and life has the two dimensional p.d.f. *f(x,y)*. Let the conventional ``extended warranty" is offered as above, with an addition W_X to the first component, and addition W_Y to the second component, whichever comes first. The two pairs of numbers (V_X, V_Y) and (W_X, W_Y) also may play a role as above. In other words, if the pair (*X,Y*) fails in the rectangle $R := [x_0, x_0 + W_X] \times [y_0, y_0 + W_Y]$ some expenses for replacements or repair will be covered by the warrantor. Chances this to happen on the area **A** = $[x_0, x_0 + V_X] \times [y_0, y_0 + V_Y]$ have probability $\iint_A f(x,y)\ dxdy$ = *P*(**A**), and then the warrantor will incur losses at certain rate c_r. If the failure happens with life components outside the area **A** bit still in the area of active warranty **R**, then chance for it has probability Q(**A,R**) = $\iint_{R\setminus A} f(x,y)\ dxdy$ and rate of uncured losses is then c_m. A minimization of the total expected losses L(V_X, V_Y) = c_r *P*(**A**) + c_m Q(**A,R**) will then define the best box **A** within the box **R** for this policy. The solution may not be unique in case that **A** is not an empty set, and does not coincide with **R.** Dependent rates on the age $c_r = c_r(x,y)$ and $c_m = c_m(x,y)$ also can be considered, and the total expected losses are given by the sum of the expected losses on two areas of warranty action L(**A**,**R**) = $\iint_A c_r(x,y) f(x,y)\ dxdy$ + $\iint_{R\setminus A} c_m(x,y) f(x,y)\ dxdy$.

By noticing that age indicators *X* and *Y* can only increase in time and using the curves of equivalent ages we see that the expected warranty expenses remain the same when item' age jumps from the (pessimistic) curve of the equivalent starting ages

$$C_0 :=\ \{(x,y);\ F_{X,Y}(x,y)\ =\ F_{X,Y}(x_0, y_0)\}$$

on the next level (pessimistic) curve of equivalent ages

$$C_W :=\ \{(x\ ,y\ );\ F_{X,Y}(x,y)\ =\ F_{X,Y}(x_0 + W_X, y_0 + W_Y)\}.$$

If p_0 and p_W be the levels of the two curves C_0 and C_W, then it is fulfilled $p_0 < p_W$. There are many rectangular boxed warranty policies which trace a possible passage from given point on the curve C_0 to a point on the curve C_V, and then to a point on the curve C_W respectively. A true warranty optimization program should look for the minimum of L(**A**,**R**) taken with respect to all admissible points (*x,y*) on the curve C_V and all admissible points (x_W, y_W) on the curve C_W, and then on all intermediate values $p_V \in [p_0, p_W]$. Let p_V^*, (x_W^*, y_W^*), and (x_V^*, y_V^*) be the arguments at which value the minimum in question is attained. Then the optimal values of the parameters of this kind of extended warranty are the pairs of numbers $(V_X = x_V^* - x_0,\ V_Y = y_V^* - y_0)$, and $(W_X = x_W^* - x_0,\ W_Y = y_W^* - y_0)$.

There are some peculiarities in this process. First of all, the growth in the values of the life characteristics, measurements (*X,Y*), can not be unilateral. It is hard to find items with multi dimensional life components where growth in one component can be frozen and only increase is in the other. On the example of cars it is not possible to drive some additional 100 thousand miles for no time. Nor it is realistic that the purchased vehicle will never be used. Therefore, the optimization we talk above is actually narrowed to some feasible sets of points. For instance, feasible points on the next level curves of equivalent ages are only those whose coordinates are grater than the coordinates of the starting point on the previous level.

The practice may use $K_{F(x_0,y_0)}[F(x_w, y_w) - F(x_0, y_0)]$ as a natural upper bound of the expected warranty expenses (here $K_{F(x_0,y_0)}$ is the price at which the used item is sold), and plan the premiums accordingly.

Example 3 Assume (X=57, Y=3.5) is the two dimensional age of the used car (see Example 2) sold for $10,000 with an extended warranty for either the next 100 thousand miles or for next 5 years, which ever comes first. The actual warranty is on certain parts (e.g. lubricated only) in the condition of prescribed maintenance. For the distribution of (X,Y) as in Example 2 this vehicle stays the curve $C_{.003009}$, and should pass on the curve $C_{..104107}$. The upper limit of the expected warranty expenses by the above advice is $1,012. If calculations are using the optimistic the survivors' curves then the item will pass from $G_{.98014}$ to $G_{.747105}$. The upper bound for the expected warranty expenses id $2,331. The second is larger since it counts losses way outside the warranty box. In fact, the chances to have this item failing in the warranty box have probability .086156. Hence the true expectation for the warranty expenses is $ 861.56.

More examples on determination of the warranty boxes under different initial ages, and estimation of the warranty expenses will be given in the talk.

Similar approach can be extended to higher dimensional life. Also, analogous situations can be observed on examples with life insurance, medical insurance, and others, as well as in guarantees related to yearnings from investments, or from multi-component portfolio.

5 Conclusions

Definitions of multi dimensional age can be given using the approaches in age comparison – the joint distribution or the joint survival function..

Measurements of the age components in used item serve as ground of evaluation of its residual value and offer a new look at the pricing of used items..

Warranty in multi dimensional age is extended over each component of the age. Use of probability distributions allows evaluation of expected warranty expenses.

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