Controllable Damage Model with Gradual Failures

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Abstract

In this paper we consider a gradually damaging system with a number of observable damage states $k \geq 2$. The system can fail by achieving a complete failure state F or in each intermediate damage state, where it can occur according to an exponential distribution with state dependent parameters. The system is equipped by a controller, that in state m: $k/2 < m < k$, signalizes about a necessity to perform a repair, where the repair time also follows exponential distribution. We consider a continuous-time Markov model for which we perform a steady-state analysis and calculate a reliability function. Cost model is developed to determine the optimal two-level control policy (m^*, k^*) that specifies the signal state and the highest damage state at minimum cost.

Author Keywords: Two-level policy, Optimal control, Markov processes, Damage model

1 Introduction

Many of present-day complex technical systems equipped by a monitoring controller can be described by the model with observable gradual damage or degradation states, where from absolutely good state to complete failure state the system passes through number of intermediate damage states, in which it usually operates with less efficiency. Different kind of preventive repair was investigated by many authors (see e.g. (1)). Some controllable degradation models were considered in (2),(3), where the optimal degradation state for the system's recovery was calculated.

In the present model we introduce two-level (m, k)-policy, that specifies a signal state where a controller signalizes about future failure and the highest number of damage states. Some times and costs are needed for the maintaining the system. Based on cost structure we derive the cost function and formulate optimization problem.

2 The Process

2.1 Description of the Model

The system states are descried by value i, $i = 0, 1, \ldots, k, F$, where $i = 0, \ldots, k$ denotes the intermediate damage states and $i = F$ denotes a complete failure state.

The state-transition-rate diagram for (m, k)-policy gradually damaging system is shown in Figure 1. When the system enters states $i = 0, \ldots, k$, two events can occur: either the transition to the further damage state with intensity λ_i or complete failure with intensity ν_i. When the system enters state m, a controller generates a signal for the preventive recovery, which takes place in exponential distributed time with parameter μ. The recovery itself occurs instantly of value m: $k/2 < m < k$. After a delay, which is exponentially distributed with parameter μ_F, the system is replaced.

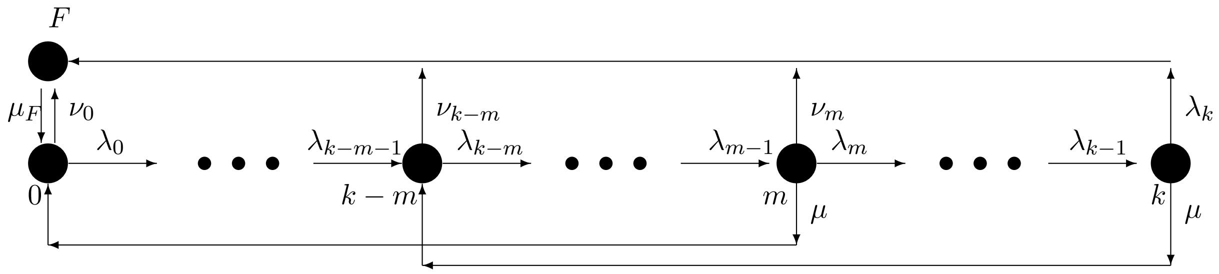

Figure 1: The transition scheme gradually damaging system under (m, k)-policy.

The steady-state probabilities π_i satisfy the following equations:

$$\begin{aligned}(\lambda_i + \delta_{i<k}\nu_i + \delta_{m\le i\le k}\mu)\pi_i &= \delta_{i=0}\,\mu_F\pi_F + \lambda_{i-1}\pi_{i-1} + \delta_{i\le k-m}\,\mu\,\pi_{i+m},\ 0 \le i \le k,\\ \mu_F\pi_F &= \sum_{0\le i\le k-1}\nu_i\pi_i + \lambda_k\pi_k. \end{aligned} \tag{1}$$

These equations admit some general solution, represented by the next statement.

Theorem 1. *The steady state probabilities for the system with controllable reliability under a two-level policy (m, k), $k \ge 2$, $k/2 < m < k$ have the form*

$$\begin{aligned}\pi_0 &= \frac{\mu_F}{\mu_F A_{mk} + B_{km}},\ \pi_F = \frac{B_{mk}}{\mu_F A_{mk} + B_{mk}}, \qquad (2)\\ \pi_i &= \frac{(q_{1i} + D_{km}C_{mi})\mu_F}{\mu_F A_{mk} + B_{mk}},\ 1 \le i \le k-m,\\ \pi_i &= \frac{q_{1i}D_{mk}\mu_F}{q_{1\,m-1}(\mu_F A_{mk} + B_{mk})},\ k-m+1 \le i \le m-1,\\ \pi_i &= \frac{q_{2i}D_{mk}\mu_F}{\mu_F A_{mk} + B_{mk}},\ m \le i \le k.\end{aligned}$$

with entries

$$\begin{aligned}\rho_{1i} &= \frac{\lambda_{i-1}}{\lambda_i + \nu_i},\ 1 \le i \le k-1,\ \rho_{2i} = \frac{\lambda_{i-1}}{\lambda_i + \nu_i + \mu},\ m \le i \le k-1,\ \rho_{2k} = \frac{\lambda_{k-1}}{\lambda_k + \mu},\\ \sigma_i &= \frac{\mu}{\lambda_i + \nu_i},\ 1 \le i \le k-m,\ q_{1i} = \prod_{j=1}^{i}\rho_{1j}\ \ \textit{with}\ \ q_{10} = 1,\ q_{2i} = \prod_{j=m}^{i}\rho_{2j},\\ C_{mi} &= q_{1i}\sum_{j=0}^{i-1}\frac{\sigma_{i-j}q_{2\,i+m-j}}{q_{1\,i-j}}\ \ \textit{with}\ \ C_{m0} = 0,\ D_{mk} = \frac{q_{1\,m-1}q_{1\,k-m}}{q_{1\,k-m} - q_{1\,m-1}C_{m\,k-m}},\\ A_{mk} &= \sum_{i=0}^{k-m}q_{1i} + D_{mk}\Bigg(\sum_{i=0}^{k-m}C_{mi} + \sum_{i=k-m+1}^{m-1}\frac{q_{1i}}{q_{1\,m-1}} + \sum_{i=m}^{k}q_{2i}\Bigg),\\ B_{mk} &= \sum_{i=0}^{k-m}\nu_i q_{1i} + D_{mk}\Bigg(\sum_{i=0}^{k-m}\nu_i C_{mi} + \sum_{i=k-m+1}^{m-1}\nu_i\frac{q_{1i}}{q_{1\,m-1}} + \sum_{i=m}^{k-1}\nu_i q_{2i} + \lambda_k q_{2k}\Bigg). \qquad (3)\end{aligned}$$

2.2 Lifetime analysis

The lifetime analysis concerns the first-passage time from state 0 to state F. Denote by T_i the lifetime of the system given initial state i . We next analyze T_i by employing a first step analysis, i.e. condition on the time of first transition out of the complete failure state F.

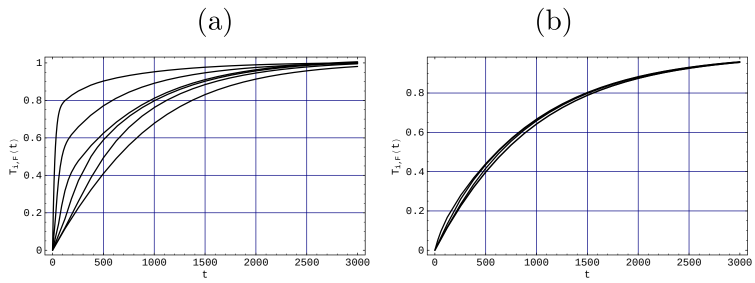

Figure 2: Conditional lifetimes $T_{i,F}(t)$ under optimal policy (m^*, k^*) versus initial state i for small repair rates (a) and large repair rates (b)

The following notation is used. Let $\tilde{f}_{i,F}(s)$ be the LT of the first-passage time from state i, $0 \le i \le k$, to state F. Using the first-step analysis we get the following result.

Theorem 2. *The transforms* $\{\tilde{f}_{i,F}(s); i \in E\}$ *satisfy the expressions*

$$\begin{aligned}
\tilde{f}_{m,F}(s) &= \frac{q_{2k}(s) + \sum_{j=0}^{k-m} q_{2\,k-1-j}(s)\tau_{2\,k-j}(s) + \sum_{j=0}^{k-m} q_{2\,k-1-j}(s)\sigma_{k-j}(s)C_{m\,k-j-m}(s)}{1 - D_{mk}(s)}, \\
\tilde{f}_{i,F}(s) &= \frac{q_{1\,m-1}(s)}{q_{1\,i-1}(s)}\tilde{f}_{m,F} + \sum_{j=0}^{m-1-i} \frac{q_{1\,m-2-j}(s)\tau_{1\,m-1-j}(s)}{q_{1\,i-1}(s)},\ 0 \le i \le m-1, \\
\tilde{f}_{i,F}(s) &= \frac{q_{2k}(s)}{q_{2\,i-1}(s)} + \sum_{j=0}^{k-i} \frac{q_{2\,k-1-j}(s)}{q_{2\,i-1}(s)}(\tau_{2\,k-j}(s) + \sigma_{k-j}(s)\tilde{f}_{k-j-m,m}(s)),\ m+1 \le i \le k, \\
\tilde{f}_{F,F}(s) &= \frac{\mu_F}{s + \mu_F}\tilde{f}_{0,F}(s)
\end{aligned} \tag{4}$$

with entries

$$\begin{aligned}
\rho_{1i}(s) &= \frac{\lambda_i}{s + \lambda_i + \nu_i},\ 0 \le i \le m-1,\ \rho_{2i}(s) = \frac{\lambda_i}{s + \lambda_i + \nu_i + \mu},\ m \le i \le k-1,\ \rho_{2k}(s) = \frac{\lambda_k}{s + \lambda_k + \mu}, \\
\tau_{1i}(s) &= \frac{\nu_i}{s + \lambda_i + \nu_i},\ 0 \le i \le m-1,\ \tau_{2i}(s) = \frac{\nu_i}{s + \lambda_i + \nu_i + \mu},\ m \le i \le k-1,\ \tau_{2k}(s) = 0, \\
\sigma_i(s) &= \frac{\mu}{s + \lambda_i + \nu_i + \mu},\ m \le i \le k-1,\ \sigma_k(s) = \frac{\mu}{s + \lambda_k + \mu}, \\
q_{1i}(s) &= \prod_{j=1}^{i} \rho_{1j}(s) \quad \textit{with} \quad q_{1\,-1}(s) = 1,\ q_{2i}(s) = \prod_{j=m}^{i} \rho_{2j}(s) \quad \textit{with} \quad q_{2\,m-1}(s) = 1, \\
C_{mi}(s) &= \sum_{j=0}^{m-1-i} \frac{q_{1\,m-2-j}(s)\tau_{1\,m-1-j}(s)}{q_{1\,i-1}(s)},\ D_{mk}(s) = \sum_{j=0}^{k-m} \frac{q_{1\,m-1}\sigma_{k-j}(s)}{q_{1\,k-j-m-1}(s)}
\end{aligned}$$

By the differentiation of the LT (4) over parameter s we calculate the mean lifetime

$$\mathbb{E}[T] = -\frac{d}{ds}\tilde{f}_{F,F}(s)\bigg|_{s=0} = \frac{1}{\mu_F \pi_F} = \frac{1}{\mu_F} + \frac{A_{mk}}{B_{mk}}.$$

The inversion of the LT $\frac{1}{s}(1 - \tilde{f}_{i,F}(s))$, $0 \le i \le k$ leads to the distribution functions of the conditional lifetimes $T_{i,F}(t)$, see e.g. Figure 2. Therefore, the reliability functions $R_{i,F} = \mathbb{P}[T_i >$

(a) (b)

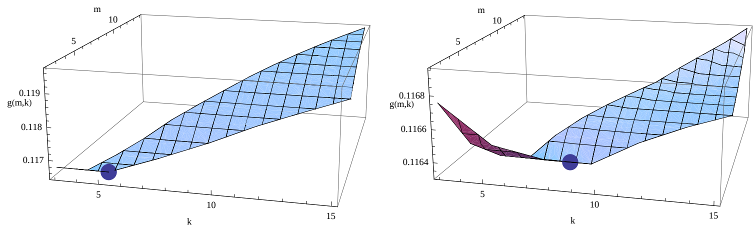

Figure 3: Long-rung average cost $g(m,k)$ versus (m,k) with labeled optimal values $(m^*,k^*) = (3,5)$ (a) and $(m^*,k^*) = (5,8)$ (b)

$t|X(0) = i]$ can now be evaluated by

$$R_{i,F}(t) = 1 - T_{i,F}(t).$$

3 Optimal (m,k)-policy

The quality of functioning of the system can be estimated by the long-run average cost function:

$$\begin{aligned} g(m,k) &= c_F\pi_F + c_e\sum_{i=0}^{k}\pi_i + c_r\sum_{i=m}^{k}\pi_i + c_f\frac{1}{\mathbb{E}[Y]} \qquad (5)\\ &= \frac{c_F B_{mk} + c_e A_{mk} + c_r\mu_F D_{mk}\sum_{i=m}^{k} q_{2i} + c_f\lambda_{m-1}D_{mk}}{\mu_F A_{mk} + B_{mk}}, \end{aligned}$$

where the first term denotes the repair cost when system is in state F, the second term denotes the cost for maintaining the system, the third term denotes the cost for system states where the preventive recovery is possible and the fourth one denotes a fixed cost for turning on the repair facility and controller. Here the period Y is a signal cycle, i.e. the length of time between two successive visits of the signal state m. The value $\mathbb{E}[Y]$ we calculate from the long-run fraction time the system is in state m

$$\mathbb{E}[Y] = \frac{1}{(\lambda_m + \nu_m + \mu)\pi_m} = \frac{\mu_F A_{mk} + B_{mk}}{\lambda_{m-1}D_{mk}\mu_F}.$$

A discrete optimization procedure is to be applied to find optimal values (m^*,k^*) of the control policy. When the state space is not large, a simple exhaustion method is quite appropriate. In Figure 2 we plot the cost function $g(m,k)$ together with the optimal values (m^*,k^*) for specified values of system parameters. Numerical results show that for the large set of the values for initial system parameters the control can essentially improve the quality of the system.

References

[1] Gertsbakh I. (2000) *Reliability theory with application to preventive maintenance.* Springer-Verlag.

[2] Rykov, V., and Efrosinin, D. (2004) On reliability control of fault tolerance units. *Proceedings MMR-2004, Santa Fe, USA.*

[3] Rykov, V., and Efrosinin, D. (2007) Risk analysis of controllable degradation model with preventive repair. *Proceedings MMR-2007, Glasgo, UK.*