Replacement Policy for Repairable System under Various Failure Types with Finite Time Horizon

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Abstract

The repairable system suffers various type of failure and each failure type has different repair cost. Assume the failure process of the system as Non-homogenous Poisson Process (NHPP). The system is replaced after it experienced a predetermined number of minimal repairs. Considering finite time horizon, the paper proposes a replacement model for the system. It firstly proves that the failure process of each type of failure also follows NHPP. Then it develops a model to estimate the total cost which covers minimal repair cost for each type of failure and system replacement cost. To obtain the numerical solution, the paper introduces a numerical approach to approximate renewal function and a nonlinear programming model is developed. A numerical example is presented eventually.

1 Notations

$\lambda(t)$	Failure rate of NHPP.
$P_i(t)$	Probability of ith failure type occurred at instantaneous time t.
$\lambda_i(t)$	Failure rate of NHPP for ith failure type.
N	Number of system failure.
X_1	Arrival time conditioned on one failure occurred during given time interval.
$f(s)$	Failure distribution density of arrival time conditioned on one failure occurred.
$N(s)$	Cumulative number of failures occurred before time s.
$N(s,t)$	Cumulative number of failures. occurred between time s and t.
$\Lambda(t)$	Mean cumulative number of failure occurred before time t.
n_i	Number of type i failure occurred.
$\mathrm{F_i(t)}$	Cumulative distribution function of type i failure.
$\mathrm{R_i(t)}$	Survival distribution function of type i failure.
m	Number of types of failure.
$\mathrm{C_i}$	Minimal repair cost for type i failure.
$\mathrm{C_r}$	Replacement cost.
C(t)	Total maintenance cost to time t.
$\mathrm{C_{min}(t)}$	Total minimal repair cost to time t.
A(t)	The non convolution part of the renewal function.
L(T)	Cost per time unit with time horizon T.
T	Time horizon.
$\mathrm{F_i}$	Value of cumulative distribution function at ith time step.
$\mathrm{A_i}$	Value of A(t) at ith time step.

2 Introduction

Most of the optimum replacement models are based on the reward renewal process with infinite time horizon. These models can obtain the analytical solution to optimum replacement time. Whereas in practice, the life length of system or the time horizon considered is finite, the models based on infinite time horizon may not be accurate. Jack (1991) presented a comparison between finite time horizon model and infinite time horizon model suitable for replacement decision and demonstrates that the cost per time unit based on finite time horizon would be 2.92% less than its corresponding cost per unit based on infinite time horizon. Castro and Alfa (2004) developed a model considering a lifetime for single unit system using age replacement policy. Other example is presented by Hartman and Murphy (2006). In some cases, the replacement models based on finite time horizon is more realistic. The paper proposes a replacement model based on finite time horizon. Assume the system subjects to NHPP[1], i.e. the system is the same as old after repair, and is suffering various types of failure. The system is replaced after it experienced a predetermined number of minimal repairs. The number of minimal repair that the system can tolerate differs at failure type. The paper proposes a methodology to determine the optimum number of minimal repairs before replacement.

3 Preliminary theory

Assume the failure of the system follows NHPP.When system failure occurs, the probability of the ith type of failure is $P_i(t)$. Ross (1996)proved the ith type of failure also follows NHPP when there are two types of failures. i.e. when parent event follows NHPP, their two child events are still NHPP. The paper considers the number of failure types(Child events) more than 2. Similar to the approach of proving for the two types of failure by Ross (1996), this Section proves that the th type of failure also follows NHPP when the number of types of failures is more than 2.

3.1 Probability of occurrence of type i failure

Condition on a failure occurs during interval [0,t], the probability of failure occurs before time s is

$$P\left\{X_1 < s\,|N(t) = 1\right\} = \frac{P\left\{X_1 < s, N(t) = 1\right\}}{N(t) = 1} \tag{1}$$

Equation(1) is rewritten to

$$P\left\{X_1 < s\,|N(t) = 1\right\} = \frac{P\left\{N(s) = 1, N(s,t) = 0\right\}}{N(t) = 1} \tag{2}$$

Due to independent increment of NHPP, the disjoint interval of NHPP is thus independent, then

$$P\left\{N(s) = 1, N(s,t) = 0\right\} = P\left\{N(s) = 1\right\}P\left\{N(s,t) = 0\right\} \tag{3}$$

Hence,

$$P\left\{X_1 < s\,|N(t) = 1\right\} = \frac{\Lambda(s)e^{-\Lambda(s)}.e^{-\Lambda(s,t)}}{\Lambda(t)e^{-\Lambda(t)}} = \frac{\Lambda(s)}{\Lambda(t)} \tag{4}$$

Therefore, the density of failure distribution conditioned on one failure occurred during interval [0,t] is

$$f(s) = \frac{\lambda(s)}{\Lambda(t)} \tag{5}$$

Therefore, the probability of the ith type of failure occur during interval [0,t] is

$$P_i = \int_0^t f(s)P_i(t)dt = \frac{\int_0^t \lambda(s)P_i(s)ds}{\Lambda(t)} \tag{6}$$

3.2 Branching of NHPP

When the system failure follows NHPP,then its each ith type of failure will also follow NHPP. It is called the branching of NHPP. The failure rate of each branches (ith type of failure) in the NHPP is:

$$\lambda_i(t) = P_i(t)\lambda(t)(\mathrm{i} = 1, 2, 3,...., \mathrm{m}) \tag{7}$$

PROOF: The joint distribution of $N_1(t), N_2(t), ..., N_n(t)$ is

$$P\left\{N_1(t) = n_1, N_2(t) = n_2, ..., N_n(t) = n_n\right\} \tag{8}$$

Which equals to

$$\sum_{n=0}^{\infty} P\left\{N_1(t)=n_1,...,N_n(t)=n_n\right\} P\left\{N(t)=n\right\} \tag{9}$$

Since $n_1+n_2+...+n_n=n$, then Equation (9) equals to

$$P\left\{N_1(t)=n_1,...,N_n(t)=n_n\left|N(t)=n\right.\right\} P\left\{N(t)=n\right\} \tag{10}$$

Where

$$P\left\{N(t)=n\right\}=\frac{\Lambda(t)^n}{n!}e^{-\Lambda(t)} \tag{11}$$

$P\left\{N_1(t)=n_1,...,N_n(t)=n_n\left|N(t)=n\right.\right\}$ follows Multinomial Distribution. Assume the total number of failure is n and the number of type *1* failures is n_1, type *2* is n_2,.... Then from the Multinomial Distribution:

$$P\left\{N_1(t)=n_1,...,N_n(t)=n_n\left|N(t)=n\right.\right\}=\frac{n!}{n_1!n_2!n_3!...n_n!}P_1(t)^{n_1}P_2(t)^{n_2}...P_n(t)^{n_n} \tag{12}$$

Where $P_1+...+P_i+...+P_n=1$ and $n_1+n_2+...+n_n=n$.

Substituting Equation(12) by Equation(6), then Equation(12) equals to:

$$P\left\{N_1(t)=n_1,N_2(t)=n_2,...,N_n(t)=n_n\right\}=\frac{\left[P_1(t)\Lambda(t)\right]^{n_1}}{n_1!}e^{-\left[P_1(t)\Lambda(t)\right]}\frac{\left[P_2(t)\Lambda(t)\right]^{n_2}}{n_2!}e^{-\left[P_2(t)\Lambda(t)\right]}...\frac{\left[P_n(t)\Lambda(t)\right]^{n_n}}{n_n!}e^{-\left[P_n(t)\Lambda(t)\right]} \tag{13}$$

Hence,

$$P\left\{N_1(t)=n_1\right\}P\left\{N_2(t)=n_2\right\}..P\left\{N_n(t)=n_n\right\}=P\left\{N_1(t)=n_1,N_2(t)=n_2,...,N_n(t)=n_n\right\} \tag{14}$$

Therefore, the various types of failure are independent from each other with mean number of failures $P_i(t)\Lambda(t)$. And

$$P_i(t)\Lambda(t)=\int_0^t \lambda(s)P_i(s)ds \tag{15}$$

Then the failure rate of ith type of failure is $\lambda(s)P_i(s)$ and

$$P\left\{N_i(t)=n_i\right\}=\frac{\left[\int_0^t \lambda(s)P_i(s)ds)\right]^{n_1}}{n_1!}e^{-\left[\int_0^t \lambda(s)P_i(s)ds\right]} \tag{16}$$

4 Optimum replacement policy model

With assumption that the repair time for the system under consideration is negligible and the system is subjected to m types of failure, each type of failure leads to minimal repair. From Formula (7), these m types of failure are independent NHPP with respective failure rates $p_1(t)\lambda(t)$,... $p_i(t)\lambda(t)$,...$p_m(t)\lambda(t)$. The replacement policy using in this paper: replace the system when it experienced n_1, or n_2,...n_i of type i failure.The paper considers the time horizon as finite.

4.1 Inter arrival time of system replacement

The replacement process of the system is a renewal process. The probability of the replacement caused by the type i failure is:

$$F_i(t)=1-R_i=1-P(M(t)<n_i)=1-\sum_{k=0}^{n_i-1}\frac{\exp\left\{-\int_0^t p_i(t)\lambda(t)dt\right\}(\int_0^t p_i(t)\lambda(t)dt)^k}{k!} \tag{17}$$

When there are m types of failure, the inter arrival time between replacements follows a distribution:

$$F(t)=1-\sum_{j=1}^{m}R_1(t)R_2(t)..R_j(t)...R_m(t) \tag{18}$$

4.2 Expected cost in finite time

The minimal repair cost for type i failure is c_i. The experienced number of repair for type i failure to time t is $N_i(t)$. System replacement cost is c_r and the experienced number of replacement to time t is N_r. Then the total cost to time t is:

$$C(t) = c_1 N_1(t) + ...c_i N_i(t)... + c_m N_m(t) + c_r N_r(t) \tag{19}$$

The failure process of type i failure is also NHPP with failure rate $\lambda(s)P_i(s)$ from Equation(7). Then the expected number of minimal repair for type i failure is:

$$E(N_i(t)) = \int_0^t \lambda(s)P_i(s)ds \tag{20}$$

Then rewrite Equation (19) to:

$$C_{\min}(t) = c_1 \int_0^t \lambda(s)P_1(s)ds...+c_i \int_0^t \lambda(s)P_i(s)ds...+c_m \int_0^t \lambda(s)P_m(s)ds = \int_0^t \{c_1P_1(s)ds...+c_iP_i(s)...+c_mP_m(s)\}\lambda(s)ds \tag{21}$$

Given the renewal process with inter arrival time *F(t)* in Equation (18), during each replacement cycle, the system subjects to minimal repair.

The cost of minimal repair follows a renewal reward process. Similar to the Formulation(10) in Reference[3], the minimal repair cost $C(t)$ to time t is

$$C(t) = A(t) + \int_0^t C(t-x)dF(x) \tag{22}$$

And

$$A(t) = c_1 \sum_{j=1}^{n_1-1} F^{(j)}(t) + ... + c_m \sum_{j=1}^{n_m-1} F^{(j)}(t) + c_r F(t) \tag{23}$$

Then the total cost per time unit is

$$L(T) = \frac{C(T)}{T} \tag{24}$$

The optimum number of minimal repair before replacement is when the *L(T)* is minimal. To obtain the solution, a nonlinear programming formulation is introduced with the objective to minimize the total cost per time unit.

$$\begin{aligned} &\min \tfrac{C(t;n_1,...n_i...,n_m)}{t} \\ &S.t. \;\; n_1 = 1,2,3,....; \\ &... \\ &\;\; n_i = 1,2,3,....; \\ &... \\ &\;\; n_m = 1,2,3,....; \end{aligned} \tag{25}$$

Where n_1,n_2,... are decision variables.

4.3 Numerical solution

Except for exponential distribution, it is not possible to obtain renewal function analytically for most distribution models, including the Weibull distribution[7]. Thus there is also hard to obtain the analytical solution to Equation (22). Whereas there are plenty of numerical approaches can be used to obtain its solution. This paper uses the approach developed by Tortorella (2005). Tortorella (2005) presents a approach using Trapezoid rule to obtain numerical solution to renewal function. In the paper, the numerical solution of $C_m(t)$ is calculated from following Equation:

$$C_m(i) = \frac{2A_i}{2-F_i} + \sum_{k=1}^{i-1} A_{i-k} \frac{F_{k+1} - F_{k-1}}{2-F_1} \tag{26}$$

Where i denotes the time of ith time step, which equals $\frac{i}{N}t$. $\frac{t}{N}$ denotes the step size which controls the accuracy of the approximation. F_i and A_i are calculated from Equation (18) and (22) respectively.

5 Numerical example

To demonstrate the model developed, we present a simple example which obtains the optimum solution using Matlab. Given failure of the system follows Power Law Process with shape and scale parameter $\alpha = 2, \beta = 1$, the system suffers two types of failure with p_1(t)= 0.3; p_2(t)=0.7. The repair costs for each type of failure are $c_1 = 10$; $c_2(t) = 20$; Table 1 presents optimum number of minimal repair for different time horizon and replacement cost. In Table 1, '*' denotes the number of repairs is unlimited, which means the system needs not to be replaced during its time horizon.

Table 1: Optimum number of minimal repairs(n_1,n_2)

	c_r=100	c_r=200	c_r=400	c_r=800
T = 4	(4,5)	(*,*)	(*,*)	(*,*)
T = 8	(3,4)	(10,16)	(10,17)	(12,18)
T = 12	(3,4)	(9,14)	(9,15)	(10,16)

6 Conclusion

The paper proposes a replacement model considering finite time horizon when the system suffers various types of failures. The paper generalizes some the existing models which consider only one type of repair. A numerical example is presented to validate its feasibility. The paper consider time horizon as finite. In practise, when the time horizon is large, the computation of obtaining optimum solution will take longer time. The time horizon can be considered as infinite and the other replacement model based on infinite time can be used.

References

[1] Barlow R.E. and Proschan F.(1965). *Mathematical Theory of Reliability*. New York: Jonh Wiley&Sons,Inc.

[2] Jack, N. (1991). Repair replacement modelling over finite time horizons *Reliab Eng Syst Saf 42*(9), 759-766.

[3] Jack, N. (1992). Costing a finite minimal repair replacement policy *Operational research society 43*(3), 271-275.

[4] Hartman J.H.,Murphy A. (2006). Finite-horizon equipment replacement analysis *IEEE Transaction on reliability 38*409-419.

[5] Castro, I.T., Alfa, A.S. (2004). Lifetime replacement policy in discrete time for a single unit system *Reliab Eng Syst Saf 84*103-111.

[6] Ross M.S. (1996). *Stochastic process*. New York: John Willey & Sons, Inc.

[7] Jiang R. (2008). A Gamma-normal series truncation approximation for computing the Weibull renewal function *Reliab Eng Syst Saf 93*616-626.

[8] Tortorella, M. (2005). Numerical solutions of renewal-type integral equations *Journal on Computing 17*(1), 66–74.