RELIABILITY MODEL FOR HIERARCHICAL SYSTEMS [1]

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Аннотация

Most of real technical systems and biological objects with sufficiently high organization are complex hierarchical partially controllable systems. In the paper for modelling and analysis of reliability of such systems the methods of decomposable semi-regenerative processes is used. Some simple example illustrate our approach.

1 Introduction and Motivation

In terms of reliability, most of technical systems and biological objects with sufficiently high organization are complex hierarchical partially controllable systems. The failures in the systems of this type arise as a result of stress accumulation of the lowest (elementary) level, which pass several stages before the full failure. These faults lead to the efficiency of the system decreasing but do not lead to the full failure of the system. The system of control (SoC) fixes these fault stages of elements and gives the signal about the system "state of health" decreasing. Accordingly to these signals appropriate mechanisms of self-regulation are "switched in", and the system is self regenerated if the process disturbing do not too deep. In the last case some outside acting is needed. It is supposed that these action being applied at time and in needed quality and quantity turn the system after some time to the normal functioning state. In another case the delay with maintenance of the system leads to the system degradation and as a result it leads to the full failure of the system. For biological systems, for instance, the neuron system play a role of controlling system, and it possesses high reliability. This means that biological objects can be treated as a complex hierarchical controllable fault tolerance reliability systems. For different technical systems there exists analogous high reliable systems of control. We focus on the survival function study, because it is the main characteristic for biological objects and complex technical systems.

In some previous papers we considered such type of models under Markov assumptions [1] - [2]. In this paper we propose a general mathematical model for the description and its survival function evaluation of complex hierarchical systems with general distributions of units life times as well as the repair times of failed units, subsystem and the whole system.

There are several approaches to model the reliability of systems with general life- and repair times distributions. However, anyhow all of them are reduced to markovization of the process that describe the system behavior. One of them was proposed by Yu.K. Belyaev [3], and consists in construction of so called linear-wise Markov processes. Another approach was developed in the works of N.P. Buslenko, I.N. Kovalenko and V.V. Kalashnikov [4] - [9], who proposed and elaborated the mathematical technique for study of so called pies-wise linear aggregative systems. The further development of this theory were done in the works [10] - [15], were the notion of Decomposable Semi-Regenerative Processes (DSRP) was proposed and methods for its investigation were developed. In this paper these methods are applied for the reliability of complex hierarchical systems investigation.

2 A General Model

Consider complex hierarchical multi-component system which is controlled and managed by a high reliable system of control (SoC), shown in the fig 1. Assume that the system is constructed from blocks and branches of several levels. Each block and the following after branches and blocks forms a hierarchical subsystem of the same type as the main one. The blocks of the lowest level will be referred to as units and they are subjected to gradual failures of its own type. For the simplicity we limit ourselves by only binary systems. We will denote by L the maximal level of units, and it is not necessary that any unit belongs to this level. Different levels units are possible. The reliability of each unit is partially controllable.

In order to specify the description its behavior we introduce vector index $\mathbf{i} = (i_1, i_2, \dots, i_{L_i})$ which determines each unit of the system as belonging to an appropriate chain of blocks at any level. Denote by

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$\mathcal{I}$ the set of these indices (and appropriate units). Then the state space of the system can be represented as $\mathcal{E} = \{\mathbf{j} = (j_{\mathbf{i}} : i \in \mathcal{I})\}$, where for any $\mathbf{i} \in \mathcal{I}$ the integer $j_{\mathbf{i}}$ represents the state of the **i**-th unit in sense of its reliability. To specify the subsystems of k-th level we will referred to it as $\mathbf{i}_j^{(k)}$ and the states of appropriate subsystem will be denoted by ${\mathbf{j}_{\mathbf{i}}}^k$.

ПРАВИТЬ РИС.

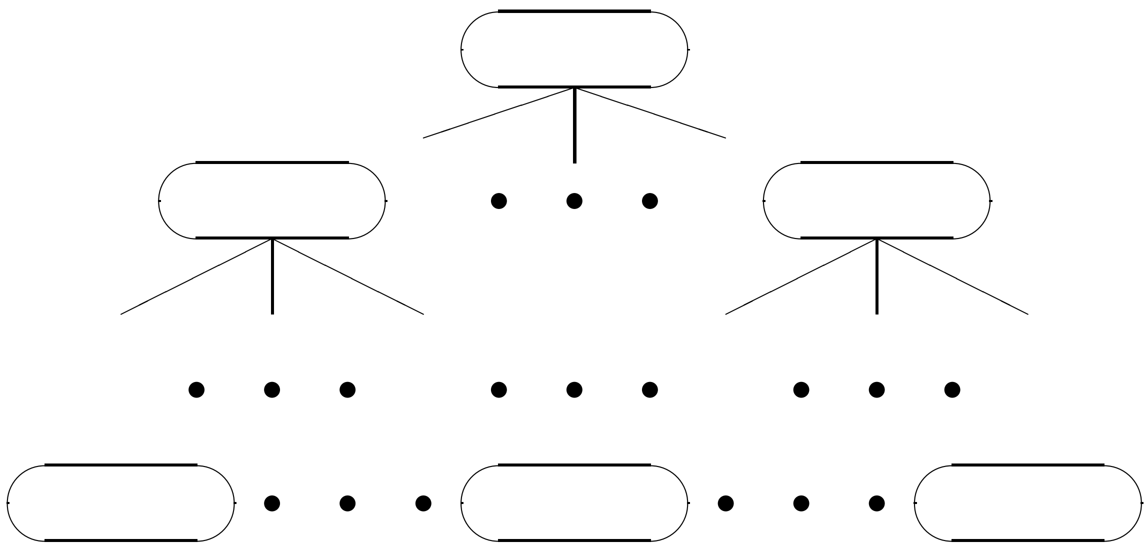

Fig. 1. A complex multi-level hierarchical system.

3 The Reliability of a General Model Investigation

Accordingly to the assumptions above we model the reliability of such a system by multi-dimensional process $\mathbf{J} = \{J_{\mathbf{i}}(t) : \mathbf{i} \in \mathcal{I}, t \geq 0\}$, with set of states $\mathcal{E}$. Similarly, the components $J_{\mathbf{i}}^{(k)}$ of the process $\mathbf{J}$ for subsystem of each level k will be considered as binary processes that takes only two states

$$J_{\mathbf{i}}^{(k)}(t) = \begin{cases} 0 & if \quad \mathbf{i}_k\text{-th subsystem of } k\text{-th level works} \\ 1 & if \quad \mathbf{i}_k\text{-th subsystem of } k\text{-th level fails} \end{cases}$$

Accordingly to the structur function of the system, the state of its units determine the state of appropriate subsytems and the whole system in sense of its reliability. After repair of failed units, subsystem or the whole system they go to the initial states. Denote by $E_{\mathbf{i}}^{(k)}$ and $\bar{E}_{\mathbf{i}}^{(k)}$ the sets of the working and failure states for **i**-th subsystem of the k-th level. Then, the working period of the whole system is given by the relation

$$W^{(0)} = \inf\{t : \ J^{(0)}(t) \in \bar{E}^{(0)}\} = \inf\{t : \ J_{\mathbf{i}_j}^{(1)}(t) \in \bar{E}_{\mathbf{i}_j}^{(1)}, \ j = \overline{1, n_1}\}$$

Analogously for each subsystem $\mathbf{i}$ of any k-th level one has

$$W_{\mathbf{i}}^{(k)} = \inf\{t : \ t \leq W^{(k)}, \ J_{\mathbf{i}}^{(k)}(t) \in \bar{E}_{\mathbf{i}}^{(k)}\} = \inf\left\{t : \ t \leq W^{(k)}, \ J_{\mathbf{i}_j}^{(k+1)}(t) \in \bar{E}_{\mathbf{i}_j}^{(k+1)}, \ j = \overline{1, n_{\mathbf{i}}^{(k)}}\right\}.$$

Therefore, considering the working period distributions of some subsystem as its life times one can investigate by the same way the reliability function of the subsystem of the next level. However, because the initial information about system is given only for the lowest level the problem should be solved beginning from the lowest level.

For the cumulative distribution function (c.d.f.) of any subsystem $\mathbf{i}_k$ of any level k calculation due to the working period definition as $W_{\mathbf{i}}^{(k)} = \inf\{t : \ J_{\mathbf{i}}^{(k)}(t) \in \bar{E}_{\mathbf{i}}^{(k)}\}$ one has

$$W_{\mathbf{i}}^{(k)}(t) = \mathbf{P}\{W_{\mathbf{i}}^{(k)} \leq t\} = 1 - \prod_{\mathbf{j} \in \bar{E}_{\mathbf{i}}^{(k)}} [1 - \pi_{\mathbf{j}}(t)]$$

where $\pi_{\mathbf{j}}(t)$ is the probability distribution for the process $\mathbf{J}^{(k)}$ to be in failure state $\mathbf{j} \in \bar{E}_{\mathbf{i}}^{(k)}$ in time t. Therefore, the problem is divided into two parts:

- to investigate the process $J_{\mathbf{i}}^{(k)}(t)$ describing behavior of any subsystem of each level;
- to find the working period $W_{\mathbf{i}}^{(k)}$ for any subsystem of each level.

Because the problems have identical solution for any subsystem they will be considered in some general construction.

4 The Subsystems Behavior Investigation

For investigation of the separate subsystem by using markovization approach consider multi-dimensional Markov process $\mathbf{Z} = (\mathbf{J}(t),\, \mathbf{X}(t))$ with general states space $\hat{\mathcal{E}} = \mathcal{E} \times R^m$, where additional components of the vector $\mathbf{X}$ describes the time passed from the entering to appropriate states. Denote by

$$\pi_{\mathbf{j}}^{(W)}(t; \mathbf{x})\mathbf{dx} = \pi_{(j_1,\dots j_m)}(t;\, x_1, \dots x_m)dx_1 \dots dx_m = \mathbf{P}\{J_i(t) = j_i,\, X_i(t) \in dx_i,\, i = \overline{1,m},\; t \le W\} \qquad (1)$$

the probability density of the state $(\mathbf{j},\, \mathbf{x})$ for the process $\mathbf{Z}$ at its separate working period W. In order to apply the methods of DSRP denote by:

- $S_1, S_2, \dots S_n, \dots$—the times of jumps of the process $\mathbf{J}$ that is coincide with times of failure or renovation elements of the system (the epochs, in which some of components of the process $\mathbf{Z}$ became equals zero). We will refer to S_n as times of k-th type renovation if $X_k(S_n + 0) = 0$;
- $\mathbf{J}_n = \mathbf{J}(S_n + 0),\;\; \mathbf{X}_n = \mathbf{X}(S_n + 0),\;\; K_n = \#\{k :\, X_k(S_n + 0) = 0\}$
- $\mathbf{j}(k) = (j_1, \dots, \bar{j}_k, \dots j_m),\; \bar{j}_k = 1 - j_k,$
- $\mathbf{x}(k) = (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots x_m),$
- $\mathbf{N} = \mathbf{N}_{\mathbf{j}}(t;\, \mathbf{dx}) = \{N_{\mathbf{j}}^{(k)}(t;\, \mathbf{dx}) :\; 1 \le k \le m\}$ renewal process with general renewal states, which components $N_{\mathbf{j}}^{(k)}(t;\, \mathbf{dx})$ are k-th type renewal process with the set of the renovation states $(\mathbf{j},\, \mathbf{dx})$,
$$N_{\mathbf{j}}^{(k)}(t;\, \mathbf{dx}) = \sum_{n \ge 0} \delta_{k,K_n} 1_{\{[0,t],\mathbf{j},\,\mathbf{dx}\}}(S_n,\, \mathbf{J}_n,\, \mathbf{X}_n);$$
- $H_{\mathbf{j}}^{(k)}(t;\, \mathbf{dx}) = \mathbf{E}N_{\mathbf{j}}^{(k)}(t, \mathbf{dx})$ —k-th tipe renewal function with the set of renovation states $(\mathbf{j},\, \mathbf{dx})$.

Denote also by $\Gamma_{j_k}(x)$ the c.d.f. of the time duration of the j_k-th component of the process $\mathbf{Z}$ stay at its state and by $\gamma_{j_k}(x)$ its hazard rate function (h.r.f.),

$$\Gamma_{j_k}(x) = \delta_{j_k,\,0}A_k(x) + \delta_{j_k,\,1}B_k(x), \quad \gamma_{j_k}(x) = \frac{\Gamma'_{j_k}(x)}{1 - \Gamma_{j_k}(x)}.$$

Remind that for p.d.f. $\Gamma(x)$ of any r.v. Γ and appropriate conditional distribution the following representations

$$\Gamma(x) = 1 - \exp\left\{-\int_0^x \gamma(w)dw\right\}; \qquad \mathbf{P}\{\Gamma > y \,|\, \Gamma > x\} = \frac{1 - \Gamma(y)}{1 - \Gamma(x)} = \exp\left\{-\int_x^y \gamma(w)dw\right\}.$$

hold. In order to give the Laplace transform (LT) $\tilde{\pi}(s;\, \mathbf{v})$ of the p.d.f. $\pi(t;\, \mathbf{x})$

$$\tilde{\pi}_{\mathbf{j}}^{(W)}(s;\, \mathbf{v}) \equiv \int_0^\infty \int_{R^m} e^{-st - \mathbf{v}'\mathbf{y}} \pi_{\mathbf{j}}^{(W)}(t;\, \mathbf{y})\, dt\, \mathbf{dy}$$

let us denote by and denote by $v = \sum_{1 \le i \le m} v_i$ and $v(k) = \sum_{i \ne k} v_i$ and introduce the following functions

$$\phi_{\mathbf{j}}^{(l)}(s,\, \mathbf{x},\, v) = \int_0^\infty \exp\left\{-(s + v)t - \sum_{1 \le i \le m} \int_{x_i}^{x_i + t} \gamma_{j_i}(\xi)d\xi\right\} \gamma_{j_l}(x_l + t)dt;$$

$$\psi_{\mathbf{j}}(s,\, \mathbf{x},\, v) = \int_0^\infty \exp\left\{-(s + v)t - \sum_{1 \le i \le m} \int_{x_i}^{x_i + t} \gamma_{j_i}(\xi)d\xi\right\} dt$$

that represents LT of the probability density one step transition of the RP $\mathbf{N}$ from the state $(\mathbf{j},\,\mathbf{x})$ to the state $(\mathbf{j}(l),\,\mathbf{y})$ in the result of l-th type renovation directly before jump, and appropriate LT of the probability densiy staying at the state $\mathbf{j}$. With these notations the following theorem holds

Teorem 1. *The LT $\tilde{\pi}_{\mathbf{j}}(s;\,\mathbf{v})$ of the p.d.f. $\pi_{\mathbf{j}}(t;\,\mathbf{x})$ can be expressed in forms*

$$\begin{aligned}\tilde{\pi}_{\mathbf{0}}^{(W)}(s;\,\mathbf{v}) &= \psi_{\mathbf{0}}(s;\,\mathbf{0},\,v) + \sum_{[k:\,\mathbf{0}(k)\in E]} \int_0^\infty \int_{R^{m-1}} e^{-su-\mathbf{v}'\mathbf{x}(k)} H_{\mathbf{0}}^{(k)}(du;\,\mathbf{dx})\psi_{\mathbf{0}}(s;\,\mathbf{x},\,v);\\ \tilde{\pi}_{\mathbf{j}}^{(W)}(s;\,\mathbf{v}) &= \sum_{[k:\,\mathbf{j}(k)\in E]} \int_0^\infty \int_{R^{m-1}} e^{-su-\mathbf{v}'\mathbf{x}(k)} H_{\mathbf{j}}^{(k)}(du;\,\mathbf{dx})\psi_{\mathbf{j}}(s;\,\mathbf{x},\,v). \qquad \blacksquare\end{aligned} \tag{2}$$

In order to get the LT

$$\tilde{h}_{\mathbf{j}}^{(l)}(s;\,\mathbf{dy}) \equiv \int_0^\infty e^{-st} H_{\mathbf{j}}^{(l)}(dt;\mathbf{dy}) \quad \text{and} \quad h_{\mathbf{j}}^{(l)}(s;\,\mathbf{v}) \equiv \int_0^\infty \int_{R^{m-1}} e^{-st-\mathbf{v}'\mathbf{y}(l)} H_{\mathbf{j}}^{(l)}(dt;\mathbf{dy})$$

of the embedded renewal functions $H_{\mathbf{j}(l)}^{(l)}(dt;\,\mathbf{dy})$ remark that in the definition of the reneawal functions the probability density one step transition from the state $(\mathbf{j},\,\mathbf{x})$ to the state $(\mathbf{j}(l),\,\mathbf{y})$ directly after jump is used. Taking into account this remark the following theorem can be proved

Teorem 2. *The LT $h_{\mathbf{j}}^{(l)}(s;\,\mathbf{v})$ of the embedded renewal functions $H_{\mathbf{j}}^{(l)}(dt;\mathbf{dy})$ for $\mathbf{j}(l)\in E$ are*

$$\begin{aligned}h_{\mathbf{e}_l}^{(l)}(s;\,\mathbf{v}) &= \phi_{\mathbf{0}}(s;\,\mathbf{0},\,v(l)) + \sum_{[k:\,\mathbf{0}(k)\in E]} \int_{R^{m-1}} \tilde{h}_{\mathbf{0}}^{(k)}(s;\,\mathbf{dx})e^{-\mathbf{v}'\mathbf{x}}\phi_{\mathbf{0}}^{(l)}(s;\,\mathbf{x},\,v(l))\\ h_{\mathbf{j}}^{(l)}(s;\,\mathbf{v}) &= \sum_{[k:\,\mathbf{j}(k)\in E]} \int_{R^{m-1}} \tilde{h}_{\mathbf{j}}^{(k)}(s;\,\mathbf{dx})e^{-\mathbf{v}'\mathbf{x}}\phi_{\mathbf{j}}^{(l)}(s;\,\mathbf{x},\,v(l))\end{aligned} \tag{3}$$

and for $\mathbf{j}(l)\in\bar{E}$

$$\tilde{\pi}_{\mathbf{j}(l)}(s;\,\mathbf{v}) = \sum_{[k:\,\mathbf{j}(k)\in E]} \int_{R^{m-1}} \tilde{h}_{\mathbf{j}}^{(k)}(s;\,\mathbf{dx})\phi_{\mathbf{j}}^{(l)}(s;\,\mathbf{x},\,v(l)). \qquad \blacksquare \tag{4}$$

The results above show that for the problem investigation in general case solution of complex integral equations is needed. From the other hand they show that the problem could be reduced to the functions $\psi_{\mathbf{j}}(.;.,.)$ and $\phi_{\mathbf{j}}(.;.,.)$ investigation.

5 Exponential case. An Example

Note that under exponential distributions of the life and repair times the functions $\phi_{\mathbf{j}}^{(l)}(s;\,\mathbf{x},\,v)$ and $\psi_{\mathbf{j}}(s;\,\mathbf{x},\,v)$ does not depend on the additional variables $\mathbf{x}$,

$$\phi_{\mathbf{j}}^{(l)}(s;\,\mathbf{x},\,v) = \frac{\gamma_{j_l}}{s+v+\gamma_{\mathbf{j}(l)}}, \qquad \psi_{\mathbf{j}}(s;\,\mathbf{x},\,v) = \frac{1}{s+v+\gamma_{\mathbf{j}}}.$$

This remark give the possibility to simplify the general equations and calculate the LT of the process $\mathbf{J}$ prbability states $\tilde{\pi}_{\mathbf{j}}^{(W)}(s) = \tilde{\pi}_{\mathbf{j}}^{(W)}(s;\,\mathbf{0})$.

Here we represent appropriate results the simplest example with only two elements in a system. Denote: $\mathbf{0}=(0,0),\ \mathbf{1}=(0,1),\ \mathbf{2}=(1,0),\ \mathbf{3}=(1,1),$. With these notations one has

$$\begin{aligned}\tilde{\pi}_{\mathbf{0}}^{(W)}(s) &= \frac{1}{s+\gamma_{\mathbf{0}}} + h_{\mathbf{0}}^{(1)}(s)\frac{1}{s+\gamma_{\mathbf{0}}} + h_{\mathbf{0}}^{(2)}(s)\frac{1}{s+\gamma_{\mathbf{0}}};\\ \tilde{\pi}_{\mathbf{k}}^{(W)}(s) &= h_{\mathbf{k}}^{(k)}(s)\frac{1}{s+\gamma_{\mathbf{k}}} \qquad (k=1,2);\\ \tilde{\pi}_{\mathbf{3}}^{(W)}(s) &= h_{\mathbf{1}}^{(1)}(s)\frac{\alpha_2}{s+\gamma_{\mathbf{1}}} + h_{\mathbf{2}}^{(2)}(s)\frac{\alpha_1}{s+\gamma_{\mathbf{2}}},\end{aligned}$$

where the functions $h_{\mathbf{j}}^{(k)}(s)$ satisfies to the equations

$$\begin{aligned} h_{\mathbf{0}}^{(k)}(s) &= h_{\mathbf{k}}^{(k)}(s)\frac{\beta_k}{s+\gamma_{\mathbf{k}}} \qquad (k=1,2); \\ h_{\mathbf{k}}^{(k)}(s) &= \frac{\alpha_k}{s+\gamma_{\mathbf{0}}}(1+h_{\mathbf{0}}^{(1)}+h_{\mathbf{0}}^{(2)}(s)) \qquad (k=1,2). \end{aligned}$$

The solutions of these equations are (for $\bar{\mathbf{k}} = (\bar{k}_1, \bar{k}_2)$)

$$\begin{aligned} h_{\mathbf{0}}^{(k)}(s) &= \frac{\beta_k(s+\gamma_{\bar{\mathbf{k}}})}{(s+\gamma_{\mathbf{0}})(s+\gamma_{\mathbf{1}})(s+\gamma_{\mathbf{2}})-\alpha_1\beta_1(s+\gamma_{\mathbf{2}})-\alpha_2\beta_2(s+\gamma_{\mathbf{1}})} \qquad (k=1,2) \\ h_{\mathbf{k}}^{(k)}(s) &= \frac{\alpha_k(s+\gamma_{\mathbf{1}})(s+\gamma_{\mathbf{2}})}{(s+\gamma_{\mathbf{0}})(s+\gamma_{\mathbf{1}})(s+\gamma_{\mathbf{2}})-\alpha_1\beta_1(s+\gamma_{\mathbf{2}})-\alpha_2\beta_2(s+\gamma_{\mathbf{1}})} \qquad (k=1,2) \end{aligned}$$

Therefore for working period LST $\tilde{w}(s) = \int_0^\infty e^{-st}W(dt) = \tilde{\pi}_{\mathbf{3}}(s)$ one has

$$\tilde{\pi}_{\mathbf{3}}(s) = \frac{\alpha_1\alpha_2(2s+\gamma_{\mathbf{1}}+\gamma_{\mathbf{2}})}{(s+\gamma_{\mathbf{0}})(s+\gamma_{\mathbf{1}})(s+\gamma_{\mathbf{2}})-\alpha_1\beta_1(s+\gamma_{\mathbf{2}})-\alpha_2\beta_2(s+\gamma_{\mathbf{1}})},$$

that is coinside with the results given by classic Markov approach.

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