Optimal stopping time under observation on non-monotone degradation

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Abstract

A problem of choice of a stopping time for an observable random process is considered. This process is interpreted as degradation of a system and determines hazard rate of the system. The loss function reflects average loss on an ansamble of identical systems. This form of loss function is typical for insurance from point of view of an insurant. Existence of an optimal stopping time follows from increasing of the loss function while a distance in time between a stopping time and a failure time increases both when the stopping time is too early, and when it is too late. The well-known result on choice of an optimal stopping time reduces the problem to solving of some equation where a value of the process at a stopping time is equal to some functional depending on the stopping time. This is a necessary condition for a Markov time to be a local minimum point of the loss function. For monotone processes the solution of this equation is unique and is the time of the first fulfillment of this condition. For non-monotone processes besides the first fulfillment time other solutions of this equation can exist. In the present work we inform about our results on a global minimum of the loss function on the set of local minima in case of non-monotone degradation. The problem is being solved for Markov processes. In this case the following alternative holds: either the first fulfillment of the equation is the optimal stopping time, or the optimal stopping time does not exist. For some partial cases it is shown how the space of parameters of the process can be divided according this alternative. For non-Markov processes it is possible to have non-trivial optimal solutions. The general conclusion is as follows: in most cases for non-monotone degradation the optimal stopping time is the trivial time of the first achievement of a local minimum (just the same as that for monotone processes).

Motivation. Ricently connection between degradation rate and hazard rate is actively being discussed (see, e.g., (1; 2; 3)). We will suppose that degradation is determined by an one-dimensional random process, and hazard rate is an increasing function of degradation. The second peculiarity of our investigation is that we consider possibility of non-monotone degradation. This phenomenon rather often takes place in practice. In science literature one can also find models with non-monotone degradation (Lehmann 2006). We are interesting in problem, when can we apply stopping rules, which are optimal in monotone case, to a system with non-monotone degradation.

Setting of the problem Let (X_t) be an observable degradation process of a system, and $h(X_t)$ be a hazard rate at an instant t, where h is a non-decreasing function. Thus we assume a time of failure ζ to be twice random, and its conditional reliability function is represented in the form

$$P(\zeta > t|\, X) = \exp\left(-\int_0^t h(X_s)\, ds\right).$$

Let us consider the simplest problem of the optimal decision theory to find an optimal stopping time. Let loss at a Markov time τ be determined as

$$F_\tau \equiv E(v(\zeta-\tau)^+ - wI(\zeta>\tau)) = C - vV_\tau - wU_\tau, \tag{1}$$

where $C = vE\zeta$ and

$$U_\tau = E\exp\left(-\int_0^\tau h(X_s)\, ds\right), \quad V_\tau = E\int_0^\tau \exp\left(-\int_0^t h(X_s)\, ds\right)\, dt. \tag{2}$$

This loss function can be interpreted as an average loss of an insurant who begins insurance at time τ before an instant ζ of the accident. He (she) pays a premium v per a time unit during the time period $\zeta-\tau$, and obtains payment w at the instant ζ. The necessary and sufficient condition for the loss function to have a local minimum at the Markov time τ is the condition that at the point $\tau(\omega)<\infty$ the function $(h(X_t(\omega)))$ $(t>0)$ intersects the level $b\equiv v/w$ up from below for almost everyone $\omega\in\Omega\equiv\mathcal{D}$ (Rasova and Harlamov 2008).

Characterization of local minimum times. For an increasing process the first hitting time of the level b is unique. A non-monotone process can intersect the level many times. Thus, it can have infinitely many local minima. Let τ_1 be the first up-crossing time of level b (assume the initial point is below than b). Let us call this time a simple local minimum time. Besides we will call τ'' a simple local minimum time if it is obtained as a shifted sum $\tau''=\tau'\dot{+}\alpha_{b,r}$, where τ' is some given simple local minimum time; $\alpha_{b,r}=\nu_{b-r}\dot{+}\mu_b$; ν_{b-r} and μ_b are the first exit times from intervals $(b-r,\infty)$ $(r>0)$ and $(-\infty,b)$ correspondingly, and the operation $\dot{+}$ means addition with a shift, i.e. $\tau_1\dot{+}\tau_2=\tau_1+\tau_2\circ\theta_{\tau_1}$ on the set $\{\tau_1<\infty\}$, where θ_t is the shift operator on Ω. Evidently, if τ' is a local minimum point then τ'' will be the same. Consequently, for any $r>0$ there exists a sequence $(\tau_{n,r})$ $(n=1,2,\ldots)$ of the simple local minimum points, which begins from the first simple time $\tau_{1,r}\equiv\tau_1$ and is constructed by the rule $\tau_{n+1,r}=\tau_{n,r}\dot{+}\alpha_{b,r}$.

An admissible mixture (mixed time) of a non-decreasing sequence of simple times (τ_n) is called to be a sum

$$\tau=\sum_{n=1}^{\infty}\tau_n I_{A_k}\quad (\bigcup_{n=1}^{\infty}=\Omega,\ A_i\cap A_j=\emptyset(i\neq j),\ A_n\in\mathcal{F}_{\tau_n}),$$

where $\mathcal{F}_{\tau_n}$ is a sigma-algebra of events preceding the Markov time τ_n. It is clear that τ is also a local minimum time. Sequences of view $(\tau_{n,r})$ for different r are connected by the following dependence.

Theorem 1 *For any positive $r_1<r_2$*

$$\alpha_{b,r_2}=\sum_{n=1}^{\infty}\alpha_{b,r_1}^n I_{A_n}\quad (A_n=\{\alpha_{b,r_1}^{n-1}\le\nu_{b-r_2}<\alpha_{b,r_1}^n\},\ \alpha_{b,r_1}^0=0).$$

The next theorem characterizes class of all local minimum times for diffusion Markov processes (X_t).

Theorem 2 *Let (P_x) $(-\infty<x<\infty)$ be a family of measures of a Markov process for which $\alpha_{b,r}\to 0$ $(r\to 0)$ and for any $r>0$ $\alpha_{b,r}^n\to\infty$ $(n\to\infty)$ P_b-a.s. Then for every local minimum time τ and $x<b$ the mixed time*

$$\tau_r=\sum_{n=1}^{\infty}\tau_{n,r}I_{A_n}\quad (A_n=\{\tau_{n-1,r}<\tau\le\tau_{n,r}\},\ \tau_{0,r}=0)$$

is well-defined and P_x-a.s. converges to τ as $r\to 0$.

Alternative of choice in case of Markov process. In two following paragraphs we assume that (X_τ) is a Markov process with a family of measures (P_x), where $P_x(X_0=x)=1$. According to the initial point of the process we denote integrals defined in (1) and (2) as $F_\tau(x)$ $U_\tau(x)$ and $V_\tau(x)$. Let us consider increments of the loss function at two neighboring simple local minimum times. We have

$$F_{\tau_{n+1,r}}(x)-F_{\tau_{n,r}}(x)=\,-w\,U_{\tau_{n,r}}(x)G_r(b),$$

where

$$G_r(b)\equiv E_b\int_0^{\alpha_{b,r}}\exp\left(-\int_0^t h(X_s)\,ds\right)(h(b)-h(X_t))\,dt \tag{3}$$

Function $G_r(b)$ determines a sign of the increment. It can be positive as well as negative. It does not depend on n. Therefore all the sequence $(F_{\tau_{n,r}}(x))$ $(n=1,2,\ldots)$ either decreases, or increases, depending on $G_r(b)$. Moreover, direction of varying of the sequence does not depend on $r>0$, as the following theorem shows.

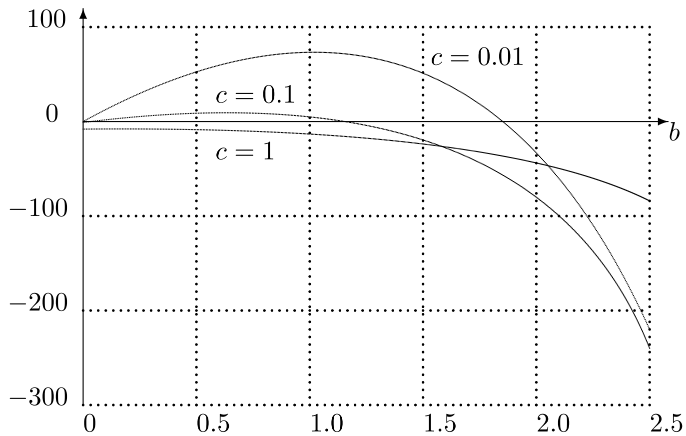

Figure 1: Derivative of $G_r(b)$ with respect to r at $r = 0$.

Theorem 3 *For any $0 \le r_1 < r_2$, such that $P_x(\alpha_{b,r_1} > 0) > 0$, the following formula holds*

$$G_{r_2}(b) = G_{r_1}(b)\left(1 - E_b\left(\exp\left(-\int_0^{\alpha_{b,r_1}} h(X_s)\,ds\right); \alpha_{b,r_1} \le \nu_{b-r_2}\right)\right)^{-1}.$$

Due to this property it seems reasonable to proclaim the following alternative of choice:
Either the first hitting time of level b (the first local minimum time) is optimal stopping time, or the optimal time does not exist.
If the problem of choice is solved only in class of all simple local minimum times a proof of the alternative follows from Theorem 3. Actually. The first solution of the alternative is fulfilled when $G_r(b) \le 0$ (increments of $F_n(x)$ are non-negative), and the second one is fulfilled when $G_r(b) > 0$ (all the increments of $F_n(x)$ are negative, i.e. the best simple time belongs to infinity). This principle happens to be extended on the whole set of local minima. It follows from Theorem 2 and the next one.

Theorem 4 *Let (X_t) be a Markov process, (τ_n) $(n = 1, \ldots, N)$ be a non-decreasing sequence of local minimum times, and τ be an admissible mixture of these times. Then for any $x < b$ the following alternative is true: either $F_{\tau_1}(x) \le F_\tau(x) \le F_{\tau_N}(x)$ $(G_r(b) \le 0)$, or $F_{\tau_N}(x) \le F_\tau(x) \le F_{\tau_1}(x)$ $(G_r(b) > 0)$.*

Partitioning of the set of levels according to the alterative of choice in case of diffusion processes. For a diffusion type process we have $G_0(b) \equiv 0$. Therefore in this case we can use a derivative of $G_r(b)$ with respect to r at point $r = 0$. Here we show an example of (X_t) a Wiener process with a positive drift. The hazard rate is determined by function $h(x) = x^+ \equiv \max\{0, x\}$. In this case $G_0'(b)$ can be evaluated with the help of solutions of two differential equations

$$\frac{1}{2}f'' + cf' - h(x)f = 0, \quad \frac{1}{2}g'' + cg' - h(x)g = -1.$$

The analytical form of these solutions under given boundary conditions at point b and on infinity can be obtained. Then the required derivative can be evaluated with the help of standard computer programs. Figure 1 shows "positive" and "negative" regions on the line under different values of drift. One can see that "positive" parts of the line take place only for very small drifts. Thus with respect to optimal stopping time Wiener process with a moderate drift behaves like a monotone process.

Example of a process with non-trivial optimal stopping time. Let us consider a renewal process N_t $(t \ge 0)$ with renewal times $(\sigma_n)_1^\infty$. With this process we connect a regenerative process (X_t): for any $t > 0$ belonging to interval $[\sigma_{n-1}, \sigma_n)$ we assume $X_t = (c/T_n)(t - \sigma_{n-1})$. This process is not Markov. So Theorem 4 is not applicable to it. Assume $h(t) = t$. In this case there exists an optimal stopping time,

Table 1: Three variants of parameters of random sets.

b	H	M
0.5	0.23	0.1
0.8	0.54	0.5
0.9	0.66	0.67

which is better than the first hitting time of the level b. This is the mixture $\tau_{opt} = \sum_{k=1}^{\infty} \tau_k I_{A_k}$, where $A_k = \overline{A}_1 \dots \overline{A}_{k-1}\, \theta_{\sigma_{k-1}}^{-1} B$, and B is a random set expressed in terms of the realized length of the current renewal interval. Non-trivial solution of the optimal problem is found in case when σ_1 is distributed according to exponential law with parameter λ. In this case

$$B = \left\{ -\frac{\sigma_1}{c} \int_b^c (u-b) e^{-\sigma_1 u^2/2c}\, du + e^{-\sigma_1 c/2}(H+M) < 0 \right\},$$

where

$$H = \frac{b}{\sqrt{2\lambda c}} \arctan \frac{b}{\sqrt{2\lambda c}},$$

and M is a solution of the equation

$$M = \max_m \int_0^m \lambda e^{-x(\lambda + c/2)} (M + H - A(x))\, dx, \tag{4}$$

where

$$A(x) = \frac{x}{c} \int_b^c e^{(1-u^2/c^2)cx/2)} (u-b)\, du.$$

Under parameters $c = 1$ and $\lambda = 1/2$ the following values for H and M were obtained for three values of levels b (see Table 1). Thus, for example, under $b = 0.9$, being up-crossed at time t, one should check the current value of expression

$$0.67 + 0.66 - T \int_{0.9}^1 e^{T(1-u^2)/2} (u - 0.9)\, du,$$

where $T = \tau/0.9$ and $\tau = t - \sigma_{N_t}$ (the time from beginning of the current interval). If this meaning is negative one should stop. If it is positive one should pass till the next local hitting time of the level 0.9. If $0 < P_b(B) < 1$ there exists a non-trivial mixed stopping time. This is the optimal Markov policy, an admissible mixture of simple local minimum times.

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