Optimization of Reliability Improvement Allocation

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Abstract

If a system does not meet its reliability requirements, the situation is usually improved by increasing the reliability of each of its components. There are many ways in which this can be done. We present a method to find an optimal combination that leads to a satisfactory system reliability with minimum cost. The algorithm is based on efficient implementation of a greedy heuristic, which avoids converging to a non-global optimum and guarantees finding a good approximation of the optimum with relatively low computational effort. The method is applicable for series, parallel and more general system configurations.

1 Introduction

If, during the course of project development, the system reliability does not meet its reliability goals, modifications become necessary. We will not discuss modification of the design, such as adding redundancy, etc., but will only consider improving the reliability of the system components. Component improvement comes at a certain cost, or more generally effort. Considering the effort required for every possible improvement, the problem is to find an optimal combination of improvements which leads to achieving the reliability goal with minimum total effort.

The literature usually refers to continuous effort functions, e.g. *Albert (1958), Agarwal & Guha (1993),* and makes the assumption that any reliability improvement is possible. Yet, in practice, one has to choose among a discrete and finite number of improvements, for example, by selecting standard components having several quality levels with corresponding fixed reliabilities and prices.

The discrete optimization problem defined above is discussed by *MIL-HDBK-338 & Agarwal (1993).* Both refer only to **series** systems. *MIL-HDBK-338* suggests the use of dynamic programming. This method yields an exact optimal solution; yet its use may be exhaustive and time consuming even when applied for small systems, as shown in the example problem given there. Agarwal suggests a greedy heuristic procedure where at each stage the reliability of a single component is increased to the next larger possible value; and the selection criterion is maximum relative increase of reliability per unit of effort. However, this procedure may sometimes yield results which are far from the optimal solution, as shown by an example in the appendix.

We present a different implementation of the greedy heuristic which avoids convergence to a non-global optimum and guarantees finding a good approximation of optimal allocation. Application of the method for series systems is presented in Sections 2 and 3, and we demonstrate by an example that it avoids convergence to the non-optimal results reached by Agarwal's method. Extension of this method for parallel systems and more general system structures is described briefly in Section 4.

2 Problem formulation

Formally the problem will be formulated as follows:

Let **R**=(R_1,R_2…R_n) be the component reliabilities vector. For each component, we consider a (discrete) number of options to improve its reliability. Let R_i^j define the achieved reliability of component i caused by the j'th kind of improvement. R_i^j will be sorted in increasing order, with R_i^0 being the current reliability. An effort function $C_i(R_i^j)$ defines the cost of any kind of improvement, with $C_i(R_i^0) = 0$. Given the system reliability goal value, r, the optimal allocation is the vector **R***=(R_1*,R_2*…R_n*) that solves the following minimization problem

(P) $\min \sum C_i(R_i)$

such that (1) $\prod(R_i) \geqq r$

(2) $R_i \in \{R_i^0, R_i^1, R_i^2 ...\}$ *for each i.*

This kind of problem is known to be NP-hard. Therefore, when dealing with large systems exact methods like dynamic programming become intractable, and a heuristic should be used.

Practically, the marginal effort – defined as the effort per one percent of reliability improvement – will increase when the component's reliability increases; which leads to the following assumption:

Assumption 1. $C_i(R_i)$ is a monotone increasing discrete convex function of R_i

Assumption 1 implies that an inverse function $R_i(C_i)$ can be defined, which is **concave**; and the problem (P) can be recast in terms of the effort function vector **C**

(P') $\min \sum C_i$

such that (1) $\prod\{R_i(C_i)\} \geqq r$

(2) $C_i \in \{C_i^0, C_i^1, C_i^2 ...\}$ *for each i.*

We find it more advantageous to use **C** rather than **R** as the decision vector, due to the linearity of the objective function and the convexity property shown ahead.

Let L(**R**) = $\sum\{-\log(R_i)\}$, Then Constraint (1) can be presented as:

(1') L(**R**) $\leqq -\log(r)$

The log transform L has the advantage of being a separable function of R_i and hence of C_i. Another important property of this function is presented in the following proposition:

Proposition 1. Under Assumption 1, L is a convex function of **C**. Therefore the region defined by (1') is convex.

Proof: Let L_i=log(R_i), then R_i=exp(L_i) is a convex function of L_i. From Assumption 1 and Theorem 4.12 in (*Avriel 1976*) C_i is a convex function of L_i. Referring to the **inverse** function, L_i is a **concave** function of C_i. Thus, -log(R_i), which is the negative of L_i, is a **convex** function of C_i. The function of L is a sum of convex functions, and therefore is also convex.

Q.E.D

3 The proposed method

We propose an iterative procedure where at each stage a single component is selected to increase its reliability from the current value R_i^j to R_i^{j+1}. For each component the marginal benefit in terms of L(**R**) per unit of effort is defined as

$$\Delta_i = \frac{-\log(R_i^{j+1}) - [-\log(R_i^j)]}{C_i^{j+1} - C_i^j}$$

(notice that Δ_i are negative). The selected component will be the one with the least value (maximum absolute value) of Δ_i. Notice that since L is a separable function, Δ_i is not affected by reliability changes of other components. This procedure is continued until the reliability constraint is satisfied.

Although this procedure does not guarantee convergence to the exact optimal solution, we expect to find a good practical approximation of the optimum, based on the following proposition:

Proposition 2. Let r' be the reliability achieved by the above heuristic algorithm, and c' the achieved total cost. c' is the global minimum of (P') if r' is inserted in constraint (1') as the value of r.
This property results from the convexity property shown above, and from the fact that both – L(**R**) and the cost function, are separable functions of C_i.

In practice, this approximate solution is expected to correspond especially well with the "true" optimum mainly when dealing with large systems having several options for each component. Notice that these are the cases where exact methods like dynamic programming become intractable.

4 Extension of the model for parallel and more general system structures

For the parallel redundant system the reliability goal constraint (1) can be presented as

$$(1'') \qquad \sum\{\log(1-R_i)\} \leq 1-r$$

Again the log transform yields a separable function of C_i, yet the functions L_i=log(1-R_i) are not necessarily convex; namely, the marginal benefit, Δ_i, does not necessarily increase when R_i increases. Thus a direct implementation of the greedy heuristic may lead to a local minimum which is far from the global one. To avoid this situation we first perform a convex "smoothing" of each of the functions L_i, by excluding points that are inside the convex hull of the function as a feasible solution. Then, we can apply the greedy heuristic to the smoothed functions.

The method can also be implemented for more general systems that can be divided into subsystems combined in series. First, we handle each subsystem separately and define an ordering of component reliability improvements "inside" each one. Then we apply our procedure for a series system, referring to each subsystem as a single component.

Appendix: Demonstrating the advantage of the proposed method

Agarwal (1992) proposed another kind of greedy heuristic, for which the properties demonstrated in Proposition 2 do not hold. Examples which indicate the inferior properties of Agarwal's method can readily be constructed.

Example 1. The reliability goal of a three component series system is 0.85. Table 1 presents the reliability cost function for each component.

Table 1: Component reliability cost functions

Component No. / **J**	**1**		**2**		3	
	R_1^j	C_1^j	R_2^j	C_2^j	R_3^j	C_3^j
0	.96	0	.95	0	.90	0
1	.97	30	.96	30	.92	40
2	.98	90	.97	65	.94	90
3	.99	180	.98	120	.96	180
4	-	-	.99	200	-	-

In this example the optimal choice - found also by our procedure, is to increase the reliability of component #3 to 0.94, leaving the other component reliabilities unchanged; which yields a reliability of 0.857 with a total cost of 90. Agarwal's procedure would increase the component reliabilities to 0.97, 0.97 and 0.92 respectively, achieving a reliability of 0.866 with a total cost of 135; which is significantly higher than the optimum.

References

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