# Gof Tests for the Inverse Gaussian Distribution


**Ioannis A. Koutrouvelis**
Department of Engineering Sciences
University of Patras
GR-26500 Patras, GREECE
*koutrouv@patras.gr*

**Alex Karagrigoriou**
Department of Mathematics and Statistics
University of Cyprus
CY-1678 Nicosia, CYPRUS
*alex@ucy.ac.cy*



## Abstract

This paper provides some new goodness of fit tests which can be used for assessing the appropriateness of both discrete and continuous distributions. The main focus is on the inverse Gaussian distribution. The first test statistic uses a standardized version of the logarithm of the empirical moment generating function in order to construct plots which are equivalent to a goodness-of-fit test whose critical points are obtained from fitted equations involving the sample size and the estimated shape of the inverse Gaussian distribution. The second test is based on measures of divergence whose asymptotic distribution is a chi-square distribution with appropriate degrees of freedom. An extensive simulation study shows that the new tests maintain good stability in level and high power across a wider range of distributions and sample sizes than other tests.


## 1 Introduction

The family of the two-parameter inverse Gaussian distribution (IG2) is one of the basic models for describing positively skewed data which arise in a variety of applications, such as repair times of an airborne communication transceiver (Chhikara and Folks, 1977), the number of visited pages per user within an internet site and quality characteristics (Sim, 2003; Moghadam and Eskandari, 2006). Most applications of IG2 are justified by the fact that the IG2 is the distribution of the first passage time in Brownian motion with positive drift. In reliability analysis and life testing it is preferable to the gamma and Weibull distributions in systems experiencing fatigue for a period of a large number of charges and then showing a decreasing failure due to their hardening. The IG2 is very similar to the lognormal distribution but the failure rate of the latter decreases eventually to zero, a property which is unrealistic in many applications. In addition, there are advantages in using the IG2 over other positively skewed models on the basis of its well-developed sampling theory which contains results analogous to some of the normal sampling theory - see e.g. Chhikara and Folks (1989).

## 2 Testing methods for the Inverse Gaussian

Testing for the IG2 is presented with the same main difficulty encountered in families which are not of the location-scale type. Namely, the distribution of the test statistic under the null hypothesis depends on the true value of the unknown shape parameter besides its dependence on the family under test, the estimation method and the sample size. A number of goodness of fit (gof) tests for exploring the appropriateness of the inverse Gaussian distribution is available. D'Agostino and Stephens (1986) categorize the gof tests in two broad groups, namely Chi-square tests and empirical distribution function (edf) tests. The most popular representative of the first class is the Pearson's chi-square test. A number of modifications of Pearson's chi-square test statistic have been proposed over the years the most notable of which are the Freeman-Tukey statistic, the modified chi-square, and the modified loglikelihood ratio statistic.

The second class of gof tests is the one based on the empirical distribution function (edf). Test statistics of this class rely on the edf generated from the observed sample to estimate the true but unknown distribution function by utilizing the ordering of the observations involved. Kolmogorov-Smirnov (KS), Anderson-Darling (A2), Cramer von Mises (CvM) and Watson (W) tests are the most popular of the tests in this class all of which are based on the Kolmogorov distance.

Another series of gof tests have been based on specific characterizations of the class of inverse Gaussian distributions by analogy to the characterizations of the family of Gaussian distributions. All of these tests have been proposed by Mudholkar.

A final class of tests looks into the problem from a different point of view and relies on parametric bootstrapping. More specifically, the omnibus tests of Henze and Klar (2002) use parametric bootstrap for the determination of the critical values of the distribution of the proposed test statistics. This type of tests though is not further pursued here since they cannot be directly compared with the other tests for which either an (approximate) asymptotic distribution under the null hypothesis or the empirical distribution function is used.

# 3 Proposed gof test and simulation studies

In this work we develop plotting procedures, equivalent to a goodness-of-fit test for testing the IG2. The procedures are based on the linearity of a standardized form of the cumulant generating function (SCGF) of the IG2 with respect to the shape parameter of the distribution and on the asymptotic behavior of an empirical counterpart of SCGF, denoted by ESCGF. Cumulant plots have been employed before by Ghosh (1996), Koutrouvelis and Canavos (1997), and Koutrouvelis et al. (2005), exploring the appropriateness of the normal, three-parameter gamma (Pearson type III) and three-parameter inverse Gaussian (IG3) distribution, respectively. More specifically, we develop plots for assessing the IG2 which are furnished with bands of 95% and 99% simultaneous confidence level around the fitted regression line of values of ESCGF on values of SCGF. The bands are based on both asymptotic results of the residuals and on finite-sample results of the supremum of standardized residuals. These plots are equivalent to 5% and 1% goodness-of-fit tests for the IG2.

We also develop goodness of fit tests based on measures of divergence. A measure of divergence is used as a way to evaluate the distance (divergence) between any two populations or functions. The most well known family of measures of divergence is the Csiszar's family known also as Csiszar's $\varphi$-divergence (Csiszar, 1963; Ali and Silvey, 1966). One of the most recently proposed measures of divergence is the Basu, Harris, Hjort, and Jones (BHHJ) power divergence between $f$ and $g$ (Basu et. al, 1998) which was recently generalized by Mattheou et. al (2009). Measures of divergence can be used in statistical inference for the construction of test statistics for tests of fit (Zografos et. al, 1990; Zhang, 2002; Chen et. al, 2004). In this work we focus on the discrete version of the BHHJ family of measures of divergence and investigate its implementation in testing statistical hypotheses for general distributions. More specifically, if we have to examine whether the data $(n_1, n_2, ..., n_m)$ come from a known multinomial distribution $M(N, P_0)$, where $P_0 = (p_{10}, p_{20}, ..., p_{m0})$, $N = \sum\limits_{i=1}^{m} n_i$ we define for any function $\Phi$ such that $\Phi(1) = \Phi'(1) = 0$ and $\Phi''(1) \neq 0$, a new statistic for the above goodness of fit test:

$$\mathrm{X}_a^2 \equiv \frac{2N\hat{d}_a}{\Phi''(1)}, \quad \hat{d}_a = d_f(P_0, \hat{P}) = \sum_{i=1}^{m} {p_{i0}}^{1+a} \Phi\left(\frac{\hat{p}_i}{p_{i0}}\right), \tag{1}$$

where $\hat{p}_i = x_i/N$. For different function $\Phi(\cdot)$ we obtain different test statistics. Continuous cases are discretized and the above test statistic is implemented with satisfactory results. For the above test statistic the following theorem is established.

**Theorem 1.** *Let $(n_1, ...., n_m) \sim M(N, P)$ with $P = (p_1, ..., p_m)$ and $p_i$, $i = 1, ..., m$ unknown parameters. Let also $W = \sum\limits_{i=1}^{m} \frac{N}{p_{i0}} \left(\frac{n_i}{N} - p_{i0}\right)^2$. Under the null hypothesis $H_0 : p_i = p_{i0}$, $i = 1, ..., m$ we have:*

(a) $\left(\min\limits_i p_{i0}^a\right) W \prec_{\mathrm{st}} \sum\limits_{i=1}^{m} \frac{N p_{i0}^a}{p_{i0}} \left(\frac{n_i}{N} - p_{i0}\right)^2 \prec_{\mathrm{st}} \left(\max\limits_i p_{i0}^a\right) W$

(b) $\mathrm{X}_a^2 - \sum\limits_{i=1}^{m} \frac{N p_{i0}^a}{p_{i0}} \left(\frac{n_i}{N} - p_{i0}\right)^2 \xrightarrow{P} 0$ *and*

(c) *the distribution of (1) is estimated to be approximately $c\mathcal{X}_{m-1}^2$, where $c = 0.5\left(\min\limits_i p_{_{i0}}^a + \max\limits_i p_{_{i0}}^a\right)$*

*where $\mathcal{X}_{m-1}^2$ is the chi-square distribution with $m-1$ degrees of freedom and $\prec_{\mathrm{st}}$ the symbol for stochastic ordering.*

We finally present the results of an extended simulation study in which the proposed goodness-of-fit test of size 5% is compared to other tests for the IG2 with gamma, Weibull, lognormal, Pareto, beta and shifted inverse Gaussian alternatives. The tests compared in the simulation study are known to have readily computable 5% critical values that do not require additional simulations. The proposed plots have linear scales and do not rely on the use of tables or values of special functions. The results show that the new test maintains stability in level and high power over a wider range of alternative distributions and sample sizes than other competing tests.

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