A Gini-Type Index for Aging/Rejuvenating Objects

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Abstract

This paper introduces a simple index that helps to assess the degree of aging or rejuvenation of repairable systems. The index ranges from -1 to 1. It is negative for the point processes with decreasing ROCOF and is positive for the point processes with increasing ROCOF.

Index Terms - aging, rejuvenation, homogeneity, non-homogeneity.

ACRONYMS

CDF	cumulative distribution function
CIF	cumulative intensity function
GPR	G-renewal process
HPP	homogeneous Poison process
NHPP	non-homogeneous Poison process PP point process
ROCOF	rate of occurrence of failures
RP	renewal process
TTF	time to failure

1 Introduction

In reliability and risk analysis, the terms *aging* and *rejuvenation* are used for describing reliability behavior of repairable as well as non-repairable systems (components). The *repairable systems* reliability is modeled by various point processes (PP), such as the homogeneous Poisson process (HPP), non-homogeneous Poisson process (NHPP), renewal process (RP), G-renewal process (GRP), to name a few. Among these PP, some special classes are introduced in order to model the so-called *improving* and *deteriorating* systems. An improving (deteriorating) system is defined as the system with decreasing (increasing) *rate of occurrence of failures* (ROCOF). It might be said that among the point processes used as models for repairable systems, the HPP (having a constant ROCOF) is a basic one.

In many practical situations, it is important to make an assessment how far a given point process deviates from the HPP, which can be considered as a simple and, therefore, strong competing model. Note that if the HPP turns out to be an adequate model, the respective system is considered as non-aging, so that it does not need any preventive maintenance (as opposed to the case, when a repairable system reveals aging).

The statistical tools helping to find out if the HPP is an appropriate model are mainly limited to statistical hypothesis testing, in which the null hypothesis is

H_0 : "The times between successive events (interarrival times) are independent and identically exponentially distributed ", and the alternative hypothesis is

H_1 : "The system is either aging or improving."

The most popular hypothesis testing procedures for the considered type of problems are the *Laplace test* (Rausand & Hoyland, 2004) and the so-called *Military Handbook test* (AMSAA, 1981). It should be noted that these procedures do not provide a simple measure quantitatively indicating how different the ROCOF of a given point process is, compared to the respective constant ROCOF of the competing HPP model.

Among such goodness-of-fit tests, one can mention the G-test, which is based on the so-called *Gini statistic* (Gail & Gastwirth, 1978). In turn, the Gini statistics originates from the so-called *Gini coefficient* used in macroeconomics for comparing an income distribution of a given country with the uniform distribution covering the same income interval. The Gini coefficient is used as a measure of income inequality (Sen, 1997). The coefficient takes on the values between 0 and 1. The closer the coefficient value to zero, the closer the distribution of interest is to the uniform one. The interested reader could find the index values sorted by countries in (List of Countries by Income Inequality, 2007) that includes the UN and CIA data.

In the following sections, we introduce a Gini-type coefficient for the repairable systems. The coefficient takes on the values between -1 and 1. The closer it is to zero, the closer the PP of interest is to the HPP. A positive (negative) value of this coefficient will indicate whether a given repairable system is deteriorating (improving). For the sake of simplicity, this Gini-type coefficient will be referred to as *GT coefficient* and denoted as C.

2 GT coefficient for repairable systems

2.1 Basic Definitions

A point process (PP) can be informally defined as a mathematical model for highly localized events distributed randomly in time. The major random variable of interest related to such processes is the number of events, $N(t)$, observed in time interval $[0, t]$. Using the nondecreasing integer-valued function $N(t)$, the point process $\{N(t), t \geq 0\}$ is introduced as the process satisfying the following conditions:

1. $N(t) \geq 0$
2. $N(0) = 0$
3. If $t_2 > t_1$, then $N(t_2) \geq N(t_1)$
4. If $t_2 > t_1$, then $[N(t_2) - N(t_1)]$ is the number of events occurred in the interval $(t_1, t_2]$

The mean value $E[N(t)]$ of the number of events $N(t)$ observed in time interval $[0, t]$ is called *cumulative intensity function* (CIF), *mean cumulative function* (MCF), or *renewal function*. In the following, term *cumulative intensity function* is used. The CIF is usually denoted by $\Lambda(t)$:

$$\Lambda(t) = E[N(t)]$$

Another important characteristic of point processes is the *rate of occurrence of events*. In reliability context, the *events* are *failures*, and the respective rate of occurrence is abbreviated to ROCOF. The ROCOF is defined as the derivative of CIF with respect to time, i.e.,

$$\lambda(t) = \frac{d\Lambda(t)}{dt}$$

When an event is defined as a failure, the system modeled by a point process with an increasing ROCOF is called *aging* (*sad, unhappy,* or *deteriorating*) system. Analogously, the system modeled by a point process with a decreasing ROCOF is called *improving* (*happy*, or *rejuvenating*) system.

The distribution of time to the first event (failure) of a point process is called the *underlying distribution*. For some point processes, this distribution coincides with the distribution of time between successive events; for others it does not.

2.2 GT Coefficient

Consider a PP having an integrable over $[0, T]$ cumulative intensity function, $\Lambda(t)$. It is assumed that the respective ROCOF exists, and it is increasing function over the same interval $[0, T]$, so that $\Lambda(t)$ is concave upward, as illustrated by Figure 1. Further consider the HPP with CIF $\Lambda_{HPP}(t) = \lambda t$ that coincides with $\Lambda(t)$ at $t = T$, i.e., $\Lambda_{HPP}(T) = \Lambda(T)$, - see Figure 1.

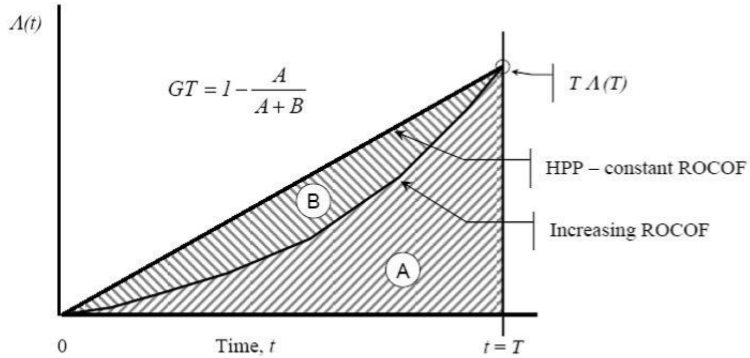

Figure 1: Graphical interpretation of GT coefficient for a point process with an increasing ROCOF.

Then, for a given time interval $[0, T]$ the GT coefficient is defined as

$$C(T) = 1 - \frac{\int_0^T \Lambda(t)dt}{0.5T\Lambda(T)} = 1 - \frac{2\int_0^T \Lambda(t)dt}{T\Lambda(T)} \tag{1}$$

The smaller the absolute value of the GT coefficient, the closer the considered PP is to the HPP; clearly, for the HPP, $C(T) = 0$. GT coefficient satisfies the following inequality: $-1 < C(T) < 1$. It is obvious that for a PP with an increasing ROCOF, the GT coefficient is positive and for a PP with a decreasing ROCOF, the coefficient is negative. One can also show that the absolute value of GT coefficient $C(T)$ is proportional to the mean distance between the $\Lambda(t)$ curve and the CIF of the HPP.

For the most popular NHPP model - the *power law* model with the underlying Weibull CDF - the GT coefficient is expressed in a closed form:

$$C(T) = 1 - \frac{2}{\beta + 1}, \tag{2}$$

where β is the shape parameter of the underlying Weibull distribution.

Some examples of applying the GT coefficient to other PP commonly used in reliability and risk analysis are given in Table 1.

Table 1: GT coefficients of some PP over time interval $[0, 2]$. Weibull with scale parameter $\alpha = 1$ is used as the underlying distribution.

Stochastic Point Process	Shape parameter of Underlying Weibull Distribution	Repair Effectiveness Factor	GT Coefficient
HPP	1	N/A	0
NHPP	1.1	1	0.05
NHPP	2	1	0.33
NHPP	3	1	0.50
RP	2	0	0.82
GRP	2	0.5	0.21

Note: the GT coefficient for RP and GRP was obtained using numerical techniques.

Repair effectiveness factor in Table 1 refers to the degree of restoration upon the failure of a repairable system; see (Kijima & Sumita, 1986), (Kaminskiy & Krivtsov, 1998). This factor equals zero for an RP, one - for an NHPP and is greater-or-equal-to zero - for a GRP (of which the RP and the NHPP are the particular cases).

As a concluding remark, we would like to note that a similar index can be used in the context of non-reparable systems (components) as well. For further details, please refer to Kaminskiy & Krivtsov (2008).

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