Recent advances in system reliability theory using signatures

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Abstract

The purpose of the talk is to show the recent advances in the representations of system reliability functions based on mixtures and signatures. The first representation obtained by Samaniego in 1985 holds only for coherent systems with independent and identically continuously distributed components. Under some symmetry assumptions we show that these representations can be extended to systems with dependent components. They can also be extended to systems with different number of components and to mixed systems, that is, to mixtures of coherent systems. To obtain these new representations, we need to do some changes in the definitions of signature vectors.

1 Signature representations of coherent systems

Let $T = \phi(X_1, X_2, \ldots, X_n)$ be the lifetime of a coherent system with independent and identically (i.i.d.) distributed component lifetimes $X_1, X_2, \ldots, X_n$ with common reliability function $\overline{F}$. The first representation of the reliability function of a coherent system with i.i.d. components with common continuous reliability function was given by Samaniego (1985). In this case its reliability function $\overline{F}_T(t) = P(T > t)$ can be written as

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i \overline{F}_{i:n}(t), \qquad (1)$$

where $\overline{F}_{i:n}(t) = P(X_{i:n} > t)$ and $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ are the order statistics obtained from the component lifetimes $X_1, X_2, \ldots, X_n$. The vector of coefficients $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ in that representation which only depends on the system structure was called the *system signature*. Moreover, the ith element in this vector is $s_i = P(T = X_{i:n})$. A survey of the applications of system signatures is given in Samaniego (2007). It is easy to see that $\mathbf{s}$ can also be computed as

$$s_i = |A_i|/n!, \text{ for } i = 1, 2, \ldots, n, \qquad (2)$$

where $|A_i|$ is the cardinality of the set A_i of permutations σ of the set $\{1, 2, \ldots, n\}$ which satisfy that $\phi(x_1, x_2, \ldots, x_n) = x_{i:n}$ whenever $x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(n)}$ (see Samaniego, 2007). Hence this can also be used as an alternative definition which only depends on the system structure.

Other useful representations for coherent systems with exchangeable components are

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t) \qquad (3)$$

and

$$\overline{F}_T(t) = \sum_{i=1}^{n} b_i \overline{F}_{i:i}(t), \qquad (4)$$

where $\overline{F}_{1:i}(t) = P(X_{1:i} > t)$ and $\overline{F}_{i:i}(t) = P(X_{i:i} > t)$ are the reliability functions of the series system lifetime $X_{1:i} = \min_{j=1,2,\ldots,i} X_j$ and the parallel system lifetime $X_{i:i} = \max_{j=1,2,\ldots,i} X_j$, respectively. The vectors of coefficients $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ which only depend on the system structure were called *minimal maximal signatures* (see Navarro, Ruiz and Sandoval, 2008), respectively. In particular, when the component lifetimes are i.i.d., representation (3) reduces to

$$\overline{F}_T(t) = \sum_{i=1}^{n} a_i(\overline{F}(t))^i = p_T(\overline{F}(t)), \tag{5}$$

where $\overline{F}$ is the common reliability function of the components and p_T is a polynomial (called *domination polynomial*) which is strictly increasing in $(0, 1)$ from $p_T(0) = 0$ to $p_T(1) = 1$. However, representations (3) and (4) are not necessarily true when the components have different distributions (see Example 5.1 in Navarro, Samaniego, Balakrishnan and Bhattacharya, 2008).

However, if $X_1, X_2, \ldots, X_n$ are not i.i.d. or they are i.i.d. with a common discrete distributions, then it is easy to see that $s_i = P(T = X_{i:n})$ depends on the distributions of $X_1, X_2, \ldots, X_n$. For example, consider $T = \min(X_1, X_2)$ with X_1, X_2 i.i.d. and $X_i = 1$ with probability $p = 1/2$ and $X_i = 0$ with probability $p = 1/2$ for $i = 1, 2$. Then $s_1 = P(T = X_{1:2}) = 1$ and $s_2 = P(T = X_{2:2}) = 1/4$. Hence, (1) does not hold. However, note that (1) for the signature vector $(1, 0)$ obtained from a similar system with i.i.d. continuously distributed components or by using (2). Hence, in this case, we could define the signature $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ of a general coherent system with lifetime T as the signature of a system with the same structure as T but with i.i.d. component lifetimes with a common continuous distribution. Equivalently, we could use (2) as the definition of signature for general coherent systems.

Navarro and Rychlik (2007) proved that (1) continues to hold when the components are dependent with a joint absolutely continuous exchangeable distribution. Recently, Navarro, Samaniego, Balakrishnan and Bhattacharya (2008) proved that (1) holds for the general exchangeable case but, in this last case, the signature should be computed using a system with the same structure and i.i.d. continuously distributed components, that is using (2). Moreover, they showed that this representation also holds for *mixed systems* which are stochastic mixtures of coherent systems (see Boland and Samaniego, 2004). However, this representation is not necessarily true when the components have different distributions (see Example 5.1 in Navarro, Samaniego, Balakrishnan and Bhattacharya, 2008).

Navarro, Samaniego, Balakrishnan and Bhattacharya (2008) also proved that if T is the lifetime of a coherent (or mixed) system with n component lifetimes extracted from an exchangeable random vector $(X_1, X_2, \ldots, X_m)$ (with $m \geq n$), then there exists a vector of coefficients $\mathbf{s}_m = (s_{1,m}, s_{2,m}, \ldots, s_{m,m})$ such that (1) holds. The vector $\mathbf{s}_m$ was called *signature of order* m of T. These vectors can be used to compare systems (in different stochastic sense) with different number of components.

Recently, Navarro, Balakrishnan and Samaniego (2008) proved that these signature representations can be extended to the case of residual lifetimes of coherent systems, that is, to $(T - t|T > t)$. For example, they showed that there exists a vector of coefficients $\mathbf{s}(t) = (s_1(t), s_2(t), \ldots, s_m(t))$ such that (1) holds for the reliability function of $(T - t|T > t)$. The vector $\mathbf{s}(t)$ was called *dynamic signature* of T. These vectors can be used to compare systems (in different stochastic sense) with different ages.

Acknowledgements

Partially Supported by Ministerio de Ciencia y Tecnología under grant MTM2006-12834 and Fundación Séneca (C.A.R.M.) under grant 00698/PI/04.

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