Planning And Scheduling Maintenance Resources In A Complex System

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Abstract

The inspection and maintenance policy is determined by the crossing of a critical threshold by an aggregate performance measure. Rather than examining the first hitting time of the level, we base our decisions on the probability that the system will never return to the critical level. The inspection policy is state dependent and we use a "scheduling function" to determine the time to the next inspection given the system state. Inspection reveals the true state of the system and allows the determination of the appropriate action, do nothing or repair, and the time of the next inspection. The approach is illustrated using a multivariate system model whose aggregate measure of performance is a Bessel process.

1 INTRODUCTION

The models derived in this paper are a natural extension of models which use the first hitting time of a critical level as a definition of failure. Here we develop a model in which the system is repaired if the probability of returning to the critical level is small (a last exit time) and it has not crossed a second level which corresponds to catastrophic failure. The intention is to maintain a minimum level of performance. The approach is appropriate when the system is subject to relatively minor repairs until it begins to degrade faster and requires major repair. This is typically the behaviour of infrastructure and large capital items. It also captures the behaviour of systems which eventually become economically obsolete and not worth repairing.

The system state is a Bessel process $R_t \in [0, \infty)$ which is transient and thus tends to increase. Transience ensures that the process will eventually escape to ∞. There are two critical levels, ξ and $\mathcal{F} > \xi$. The system is repaired if on inspection it has a small probability of returning to ξ, and suffers a catastrophic failure if it reaches $\mathcal{F}$. The time to inspection and repair is determined by a scheduling function (**?**) which gives the time until the next action as a function of the current state.

The threshold ξ defines the repair actions and is incorporated in a maintenance function ρ. The actions are determined by the probability that the process has escaped from $[0, \xi)$ and $\mathcal{F}$ defines the failure of the system and hence its replacement. The sequence of failure and replacement times $G^0_{\mathcal{F}}$ constitutes a renewal process. This embedded renewal process is used to derive the expected cost per unit time over an infinite time horizon (for the periodic inspection policy) and the total expected cost (for the non-periodic inspection policy). The costs are optimized with respect to the system parameters.

1.1 *Modelling degradation*

The system state is described by an N-dimensional Wiener process

$$\mathbf{W}_t = \underline{\mu} t + \sigma \mathbf{B}_t,\ \mathbf{W}_0 = \underline{0},\ \underline{\mu} = [\mu_1, \ldots, \mu_N]^T,\ \mathbf{B}_t = \left[B_t^{(1)}, \ldots, B_t^{(N)}\right]^T$$

where $B_t^{(i)}$ is a standard Brownian motion.

Decisions are based on a performance measure $R_t = \|\mathbf{W}_t\|_2$, the L_2-norm of $\mathbf{W}_t$. Without loss of generality we assume that $\sigma = 1$. R_t is the radial norm of a drifting Brownian motion starting at the origin, a Bessel process $Bes_0(\nu, \mu)$ starting at the origin with parameter ν and drift μ (**?**) where

$$\nu = \frac{1}{2}N - 1, \qquad \mu = \|\underline{\mu}\|_2$$

Because a Bessel process always begins at the origin (**?**; **?**; **?**) we handle repair by adjusting the thresholds. Repair restarts the process from the origin (in R^N) and lowers the threshold to that remaining distance.

2 PERIODIC INSPECTIONS

2.1 *Features of the model*

2.1.1 *Model assumptions*

a) inspection is at fixed intervals τ; b) inspections are perfect and instantaneous; c) the system state is known only at inspection or failure; d) the system starts from new, at $t = 0$; e) the thresholds are $\mathcal{F}$ and $\xi < \mathcal{F}$; f) each inspection incurs a fixed cost c_i; g) catastrophic failure is at the first hitting time of $\mathcal{F}$; h) replacement is instantaneous with a new system at cost C_f; i) maintenance actions incur a cost C_r; j) the transition density for R_t starting from x is $f_\tau^x(y) \equiv f(y|x,\tau)$.

2.1.2 *Settings for the model*

The decisions are based on the last exit time from the critical threshold ξ

$$H_\xi^0 = \sup_{t\in\mathbb{R}^+} \{R_t \leq \xi \mid R_0 = 0\} .$$

Because H_ξ^0 is not a stopping time, we work with the probability $\mathbb{P}[H_{\xi-x}^0 \geq \tau]$ of not returning to the level ξ before the next inspection.

The catastrophic failure time is a stopping time

$$G_\mathcal{F}^0 = \inf_{t\in\mathbb{R}^+} \{R_t = \mathcal{F} \mid R_0 = 0\} .$$

with density $g_\mathcal{F}^x$.

Inspection at time $t = \tau$ reveals the system's state R_τ and the level of maintenance depends on whether the system has failed $G_\mathcal{F}^0 \leq \tau$ or is still working $G_\mathcal{F}^0 > \tau$. Replacement is determined by the first hitting time of threshold $\mathcal{F}$.

Maintenance is modelled by a function ρ specifying the decreases in the threshold values. The maintenance function depends on the probability of return to ξ:

$$\rho(x) = \left\{ \begin{array}{ll} x, & \mathbb{P}[H_{\xi-x}^0 \leq \tau] \leq 1-\epsilon \\ kx, & \mathbb{P}[H_{\xi-x}^0 \leq \tau] > 1-\epsilon\,, \end{array} \right. \quad 0 < \epsilon < 1\,, k \in [0,1].$$

Standard models can be recovered: $\epsilon = 0$ no maintenance; $k = 1$ minimal repair; and $k = 0$ perfect repair.

The cost function depends on the amount, ρ, by which the threshold values are decreased

$$C_r(x) = \left\{ \begin{array}{ll} 0, & \mathbb{P}[H_{\xi-\rho(x)}^0 \leq \tau] \leq 1-\epsilon \\ C_{rep}, & \mathbb{P}[H_{\xi-\rho(x)}^0 \leq \tau] > 1-\epsilon \end{array} \right.$$

2.1.3 *The framework*

There are two possibilities, the system fails $G_\mathcal{F}^0 \leq \tau$ or is still working $G_\mathcal{F}^0 \leq \tau$ after crossing ξ. At inspection time t_1 and before any maintenance action, the performance measure is $R_{t_1} = x$. Maintenance lowers the threshold values $\xi \mapsto \xi - \rho(x)$ and $\mathcal{F} \mapsto \mathcal{F} - \rho(x)$, so considering the next interval

1. $G_{\mathcal{F}-\rho(x)}^0 > \tau$: the system survives until the next planned inspection in τ units of time at $t_1 + \tau$ with cost C_i. The cost of repair at this next inspection is $C_r(\rho(x))$. The performance measure at time $t_1 + \tau$ is R_τ^0 and determines the reduction in the thresholds.

2. $G_{\mathcal{F}-\rho(x)}^0 \leq \tau$: the performance measure hits the threshold $\mathcal{F} - \rho(x)$ before the inspection at $t_1 + \tau$ and is instantaneously replaced with cost of failure C_f. These failure times form a renewal process.

Each cycle consists of a sequence of occurrences of case 1 and ends with case 2 as the system fails and is replaced.

2.2 *Optimal periodic inspection policy*

2.2.1 *Expected cost per cycle*

If $R_\tau = x$ at time τ^- an inspection prior to any maintenance, we set $R_{\tau^+} = x$ and the threshold values to $\mathcal{F} - \rho(x)$ and $\xi - \rho(x)$ at τ^+ just after the action. A renewal type argument yields the cost of inspection and maintenance per cycle given that at $R_\tau = x$.

$$V_\tau^x = C_f \mathbf{1}_{\{G^0_{\mathcal{F}-\rho(x)} \le \tau\}} + \left[C_i + C_r(x) + V_\tau^{R^0_\tau}\right] \mathbf{1}_{\{G^0_{\mathcal{F}-\rho(x)} > \tau\}}$$

where $V_\tau^{R^0_\tau}$ is the future cost restarting from the renewed state 0.

Taking the expectation:

$$v_\tau^x = \mathbb{E}\left[V_\tau^x\right] = A + B \tag{1}$$

The expected cost can be written

$$v_\tau^x = Q\left(x\right) + \lambda\left(x\right) \int_0^{\mathcal{F}-\rho(x)} v_\tau^y f_\tau^0\left(y\right) dy \tag{2}$$

$$\text{with } \lambda\left(x\right) = 1 - \int_0^\tau g^0_{\mathcal{F}-\rho(x)}\left(y\right) dy \qquad Q\left(x\right) = \left(1 - \lambda\left(x\right)\right) C_f + \lambda\left(x\right)\left\{C_i + C_r\left(\rho(x)\right)\right\} \tag{3}$$

2.2.2 *Expected length of a cycle*

The expected length of a cycle, l_τ^x, is obtained similarly. The length of a cycle L_τ^x is

$$L_\tau^x = G^0_{\mathcal{F}-\rho(x)} \mathbf{1}_{\{G^0_{\mathcal{F}-\rho(x)} \le \tau\}} + \left[\tau + L_\tau^{R^0_\tau}\right] \mathbf{1}_{\{G^0_{\mathcal{F}-\rho(x)} > \tau\}}$$

where $L_\tau^{R^0_\tau}$ is the length of a cycle restarting in state 0. The expected value is

$$l_\tau^x = P\left(x\right) + \lambda\left(x\right) \int_0^{\mathcal{F}-\rho(x)} l_\tau^y f_\tau^0\left(y\right) dy \tag{4}$$

with λ defined in (3) and

$$P\left(x\right) = \int_0^\tau y g^0_{\mathcal{F}-\rho(x)}\left(y\right) dy + \tau\lambda\left(x\right) \tag{5}$$

2.2.3 *Expected cost per unit time*

A standard renewal reward argument gives the cost per unit time $\mathcal{C}_\tau^x = \dfrac{v_\tau^x}{l_\tau^x}$ with expressions for v_τ^x, l_τ^x given in (1) and (4) respectively.

2.2.4 *Obtaining solutions*

The density, $g^0_\mathcal{F}$, for the first hitting time of a Bessel process with drift is known only through its Laplace transform (**?**; **?**).

$$\mathbb{E}[e^{-\frac{1}{2}\beta^2 G^0_\mathcal{F}}] = \left(\frac{\sqrt{\beta^2+\mu^2}}{\mu}\right)^\nu \frac{I_\nu\left(\mu\mathcal{F}\right)}{I_\nu\left(\mathcal{F}\sqrt{\beta^2+\mu^2}\right)}, \quad \nu > 0$$

Solutions to (2) were obtained by numerical inversions of the Laplace transform (**?**).

The Volterra equations (2) and (4) are reformulated as Fredholm equations

$$v_\tau^x = Q\left(x\right) + \lambda\left(x\right) \int_0^\mathcal{F} K\left\{x,y\right\} v_\tau^y \, dy \qquad l_\tau^x = P\left(x\right) + \lambda\left(x\right) \int_0^\mathcal{F} K\left\{x,y\right\} l_\tau^y \, dy$$

with Q, λ as in (3), P as in (5) and $K\left\{x,y\right\} = \mathbf{1}_{\{y \le \mathcal{F}-\rho(x)\}} f_\tau^0\left(y\right)$. They are solved numerically using the Nystrom routine with a Gauss-Legendre rule.

The optimal period of inspection and repair threshold can then be determined as:

$$\left(\tau^*, \xi^*\right) = \underset{(\tau,\,\xi) \in \mathbb{R}^+ \times [0,\mathcal{F}]}{\operatorname{argmin}} \left\{\mathcal{C}_\tau^0\right\}$$

3 NON-PERIODIC INSPECTIONS

3.1 *Features of the model*

The extension to non-periodic inspection policies shares many features with the periodic policy described in 2.1. The complexities of a dynamic programming formulation are avoided by introducing a scheduling function $\tau = m(x)$ which determines the time τ to the next inspection based on the observed system state x. The scheduling function develops the sequence of inspections in the following way: an inspection at τ_i reveals $R_{\tau_i} = x$, the repair is $\rho(x)$ and the next inspection is scheduled at $m(\rho(x))$.

To avoid a circular definition maintenance function ρ employs the performance measure just before repair

$$\rho(x) = \begin{cases} x, & \mathbb{P}[H^0_{\xi-x} \leq m(x)] \leq 1-\epsilon \\ kx, & \mathbb{P}[H^0_{\xi-x} \leq m(x)] > 1-\epsilon \end{cases}$$

with $0 < \epsilon < 1$, $k \in (0, 1]$.

The next inspection is scheduled at $m(\rho(x))$ units of time with cost $C_r(\rho(x))$, where

$$C_r(\rho(x)) = \begin{cases} 0, & \mathbb{P}[H^0_{\xi-\rho(x)} \leq m(\rho(x))] \leq 1-\epsilon \\ C_{rep}, & \mathbb{P}[H^0_{\xi-\rho(x)} \leq m(\rho(x))] > 1-\epsilon \end{cases}$$

3.2 *Expected total cost*

The expected total cost in the non-periodic case may be deduced with the use of the scheduling function.

$$\begin{aligned} V^x &= \left[C_f + V^0\right] \mathbf{1}_{\text{fails given } \rho(x)\}} + \left[C_i + C_r(x) + V^{R^0_{m(\rho(x))}}\right] \mathbf{1}_{\text{survives given } \rho(x)\}} \\ &= \left[C_f + V^0\right] \mathbf{1}_{\{G^0_{\mathcal{F}-\rho(x)} \leq m(\rho(x))\}} + \left[C_i + C_r(x) + V^{R^0_{m(\rho(x))}}\right] \mathbf{1}_{\{G^0_{\mathcal{F}-\rho(x)} > m(\rho(x))\}} \end{aligned}$$

Taking the expectation $v^x = \mathbb{E}[V^x]$ may be re-arranged as

$$v^x = Q(x) + (1 - \lambda(x))\, v^0 + \lambda(x) \int_0^{\mathcal{F}} K\{x, y\}\, v^y dy \tag{6}$$

with $\lambda(x) = 1 - \int_0^{m(\rho(x))} g^0_{\mathcal{F}-\rho(x)}(y)\, dy$, $Q(x) = (1 - \lambda(x))\, C_f + \lambda(x) \{C_i + C_r(\rho(x))\}$, and $K\{x, y\} = \mathbf{1}_{\{y \leq \mathcal{F}-\rho(x)\}} f^0_{m(\rho(x))}(y)$

3.3 *Obtaining solutions*

While (6) contains v^x and v^0 because we need only the value of v^0 we change the equation to

$$v^x = Q(x) + (1 - \lambda(x))\, v^x + \lambda(x) \int_0^{\mathcal{F}} K\{x, y\}\, v^y dy \tag{7}$$

and solve for v^x as for the periodic model as in (**?**). The solution is then obtained by setting $x = 0$.

4 NUMERICAL RESULTS

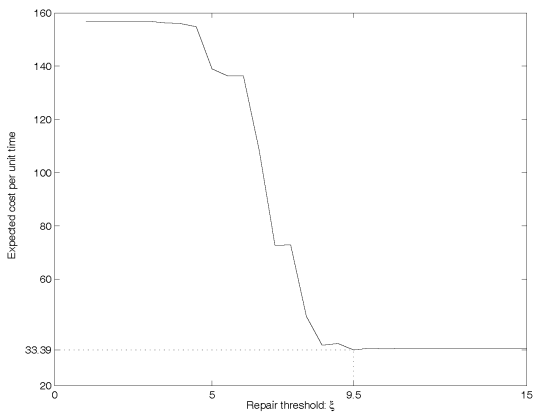
4.1 *Periodic inspection policy*

Numerical results are based on $Bes_0(0.5, 2)$ as R_t and a failure threshold $\mathcal{F} = 12$. The choice for the costs and the maintenance function's parameters are $(C_i, C_{rep}, C_f) = (50, 200, 500)$ and $(k, \epsilon) = (0.9, 0.5)$. They are varied to investigate the behaviour of the model. The choices were chosen arbitrarily to show some important features of both the inspection and maintenance policies. In view of the limited space we present only partial results.

4.1.1 *The influence of the costs*

The response of the model to costs is examined with costs $C_i \in \{0.5, 50, 500\}$, $C_{rep} \in \{2, 200, 2000\}$ and $C_f \in \{5, 500, 5000\}$.

Changing C_{rep} affects the optimal period of inspection and gives higher values for the optimal repair threshold. The higher threshold ξ^* reduces the frequency of repairs hence reducing costs. The optimal strategy is driven by the repair threshold which determines the frequency of maintenance and thus the optimal expected total cost.



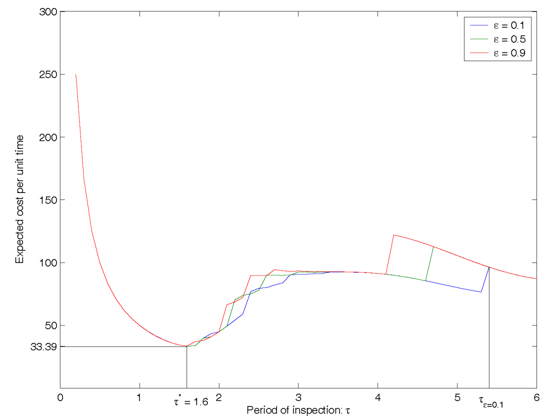


Figure 1: Effect of parameter ξ on ${C_\tau^0}^*$ with $(a, \epsilon, \tau^*) = (0.9, 0.5, 1.6)$.

Figure 2: Effect of parameter ϵ on the optimal solution ${C_\tau^0}^*$ with parameters $(k, \xi^*, \tau^*) = (0.9, 9.5, 1.6)$.

4.1.2 *Investigating the maintenance actions*

The maintenance function considered has parameters $(k, \epsilon) = \left(\frac{9}{10}, \frac{1}{2}\right)$

$$\rho(x) = \begin{cases} x, & \mathbb{P}[H^0_{\xi-x} \leq \tau] \leq \frac{1}{2} \\ 0.9x, & \mathbb{P}[H^0_{\xi-x} \leq \tau] > \frac{1}{2} \end{cases}$$

The effects of parameters ξ, k and ϵ on the model with $(C_i, C_{rep}, C_f) = (50, 200, 500)$ are examined and gave optimal parameters $(\tau^*, \xi^*) = (1.6, 9.5)$.

(i) **The repair threshold:** The optimal solution depends strongly on ξ when other parameters remain fixed as is shown in figure 1.

(ii) **Level of repair:** The level of repair increases uniformly from perfect repair with $k = 0$ to minimal repair with $k = 1$. The cycle length decreases and the cost increases as k increases.

(iii) **Repair Parameter** Repair is determined by the parameter ϵ and the probability

$$\mathbb{P}[H^0_{\xi-x} \leq \tau] \, .$$

The different values for ϵ, 0.1, 0.5 and 0.9 reflect the decision maker's attitude towards repair. Values close to 1 corresponds almost certain repair and as the value decreases to 0 repair occurs less frequently, a riskier position. The results show that only the threshold responds to ϵ.

ϵ	0.1	0.5	0.9
ξ^*	8	9.5	11
τ^*	1.6	1.6	1.6
${C_\tau^0}^*$	33.39	33.39	33.39

Table 1: Optimal parameters for different ϵ

The model adapts itself to the decision maker's attitudes to repair (the value of ϵ) by moving the optimal repair thresholds. As ϵ increases repairs will be considered more often but ξ^* increases to restrain the frequency of repairs. The optimal expected cost per unit time remains constant in the three cases studied. Figure 2 clearly shows that this is not the case for inspection periods $\tau \in (\tau^*, \tau_{\epsilon=0.1}]$, where $\tau_{\epsilon=0.1}$ satisfies

$$\forall\, t > \tau_{\epsilon=0.1} : \mathbb{P}\left[H^0_\xi < t\right] > 1 - 0.1$$

For most values in this interval, the expected cost per unit time increases with ϵ: the model penalizes a costly strategy that favors too many repairs. For a period of inspection greater than $\tau_{\epsilon=0.1}$, the expected costs per unit time are identical since in all three cases the approach towards repair is similar: the system will be repaired with certainty

$$\begin{aligned}\mathbb{P}\left[H_\xi^0 < t\right] > 0.9 &\Rightarrow \mathbb{P}\left[H_\xi^0 < t\right] > 0.5 \\ &\Rightarrow \mathbb{P}\left[H_\xi^0 < t\right] > 0.1\,.\end{aligned}$$

4.2 *Non-periodic inspection policy*

The results are obtained using $Bes_0\,(0.5, 2)$ for R_t and with f$\mathcal{F} = 5$. The different costs were explored with maintenance function parameters $(C_i, C_{rep}, C_f) = (50, 100, 200)$,$(k, \epsilon) = (0.1, 0.5)$.

Different inspection policies are obtained through the choice of three scheduling functions m_1, m_2 and m_3 modelled on the approach in (**?**).

$$m_1[x \mid a, b] = \max\{1, a - \frac{a-1}{b}x\}$$

$$m_2[x \mid a, b] = \begin{cases} \dfrac{(x-b)^2}{b^2}(a-1)+1, & 0 \leqslant x \leqslant b \\ 1, & x > b. \end{cases}$$

$$m_3[x \mid a, b] = \begin{cases} -\left(\dfrac{\sqrt{a-1}}{b}x\right)^2 + a, & 0 \leqslant x \leqslant b \\ 1, & x > b \end{cases}$$

All the functions decrease from a to 1 on the interval $[0, b]$ and then remain constant at 1. The different shapes reflect different attitudes to repair, m_1 tends to give longer intervals initially and m_3 gives shorter intervals more rapidly.

The optimal solution is determined by the optimal parameter values (a^*, b^*) rather than the solution of a more complex dynamic programming problem to determine a policy $\Pi = \{\tau_1, \tau_2, \ldots, \tau_n, \ldots\}$.

4.2.1 *The optimal maintenance policy*

Optimal solutions for a range of thresholds and scheduling functions were obtained in all cases. The solutions are not particularly sensitive to (a, b), but range over several orders of magnitude as the threshold ξ varies.

4.2.2 *The influence of costs*

We take an example with scheduling function m_1 and $\xi = 3$. The optimal parameters (a^*, b^*) and associated total cost show that as C_i increases, the optimal values of a and b increase making inspection less frequent when the cost of inspection increases.

4.2.3 *Investigating the maintenance actions*

(i) **Level of repair** The optimal cost for the three optimal inspection scheduling functions and $k \in [0, 1]$ and repair threshold $\xi = 3$ are shown in Figure 3. In all three cases the expected total cost increases with k implying a reduction in the amount of maintenance undertaken on the system at each repair. The system will therefore require more frequent repairs or will fail sooner implying an increase in the total expected cost value.

(ii) **Attitude to repair:** The attitude of the decision maker towards repair is reflected in the parameter $\epsilon \in [0, 1]$. We determined optimal expected costs with optimal parameters $\epsilon = 0.1, 0.5, 0.9$ and $\xi = 3$. Changes in ϵ induce changes in the optimal inspection policy and the resulting optimal expected total cost.

5 SUMMARY

The aim of the models derived and investigated in the present paper include adaptive repair and catastrophic failure. A second threshold $\mathcal{F}$ accounts for catastrophic failure. The threshold ξ is incorporated

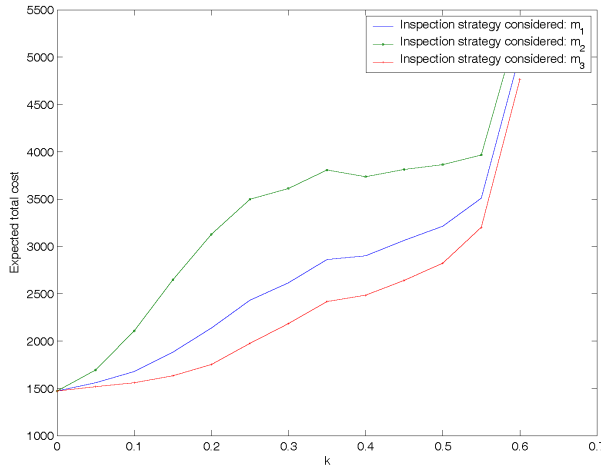


Figure 3: Optimal expected total cost as a function of k for the three inspection strategies.

in the repair function ρ as the last exit time from the interval $[0, \xi)$. The repair decision depends on the probability of occurrence of this last exit time before the next inspection. The models proposed hence include both a stopping time (the first hitting time) and a non-stopping time (the last exit time). The probability density function of the first hitting time for a Bessel process with drift being not known explicitly, the expression for the expected total cost was solved numerically (a numerical inversion of the Laplace transform of the first hitting time's density function was required).

The numerical results revealed a strong influence of the threshold's value ξ and parameter k on both the optimal period of inspection and the optimal expected cost per unit time. Letting parameter ϵ vary produced changes in the optimal repair threshold only, suggesting that the optimal strategy aims at keeping a relatively constant frequency of repairs.

References