Reliability of Markovian Missions

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Abstract

We consider a mission-based reliability system that is designed to perform missions consisting of a random sequence of phases or stages with random durations. The mission process is described by a Markov process and the deterioration or aging of the system is described by a finite state Markov process whose parameters depend on the mission process. We discuss models and tools to compute system, mission, and phase reliabilities. Explicit computational formulas are derived and an illustration is provided.

1 Introduction

We consider a system that performs a Markovian mission with a random sequence of phases which have exponentially distributed durations. Moreover, there is discrete deterioration or aging during each phase which is further modulated by the mission process. The analysis is on various measures of reliability. In particular, we focus on system reliability (probability of survival until a pre-determined time), mission reliability (probability of completing a pre-determined number of phases), and phase reliability (probability of completing a critical phase until a pre-determined time). Our setting is simpler than the semi-Markov model in Çekyay and Özekici (2008) and the main incentive behind this simplification is the desire to find more computationally tractable results.

The analysis and the structure of mission-based systems are quite similar to systems working under random environments. Random environments are generally used as a factor of variation in the failure structure of the components. An interesting model was introduced by Çınlar and Özekici (1987) where stochastic dependence is introduced by a randomly changing common environment that all components of the system are subjected to. The approach used to model the deterioration of the system in this paper is also related to condition-based maintenance systems. We refer the interested reader to Wang et al. (2009) and the references cited in this paper for more information on such models.

In Section 2, we describe the mission and aging processes. The results of the reliability analysis are presented in Sections 3.1, 3.2, and 3.3. Numerical illustrations of the results given in the previous sections are given in Section 4.

2 Mission and Age Processes

The mission process $X = \{X_t : t \geq 0\}$ is a Markov process with a finite state space E, infinitesimal generator H, transition probability matrix Q, and transition rate vector μ. We suppose that the deterioration level or age of the system takes values in some finite set $F = \{0, 1, \cdots, M\}$ where 0 stands for a brand new system and M represents system failure. The age process of the system is $A = \{A_t : t \geq 0\}$ with state space F and we assume that A is modulated by X. The age process follows a Markov process with state space F, generator G_i, transition probability matrix P_i, and transition rate vector λ_i during phase i. We also assume that M is an absorbing state for all i unless otherwise specified, i.e., $P_i(M, M) = 1$ and $\lambda_i(M) = 0$. The age process A does not really measure "real age" in time, but it indicates the deterioration level the system.

We use the bivariate Markov process $(X, A) = \{(X_t, A_t) : t \geq 0\}$ in the analysis. It is clear that (X, A)

is also a Markov process with state space $E \times F$ and infinitesimal generator

$$G(i,a;j,b) = \begin{cases} G_i(a,b) & \text{if } a \neq M, i = j, a \neq b \\ H(i,j) & \text{if } a \neq M, i \neq j, a = b \\ G_i(a,a) + H(i,i) & \text{if } a \neq M, i = j, a = b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

for every $i, j \in E$ and $a, b \in F$.

To simplify the notation, for any event C and any random variable Z, we will let $P_{ia}(C) = P(C|X_0 = i, A_0 = a)$, and $E_{ia}[Z] = E[Z|X_0 = i, A_0 = a]$. We let $\overline{1}$ denote the column vector with all entries being equal to 1.

3 Reliability Computations

The lifetime of the system is

$$L = \inf\{t \geq 0 : A_t = M\}$$

which is clearly a first passage time of the Markov process (X, A).

3.1 System Reliability

Since L is the first passage time of A to the absorbing state M, it is clear that the system reliability function is

$$P_{ia}\{L > t\} = P_{ia}\{A_t \neq M\} = \sum_{j\in E}\sum_{b=0}^{M-1} P_{ia}\{(X_t, A_t) = (j,b)\}.$$

It is well-known that

$$P_{ia}\{(X_t, A_t) = (j,b)\} = e^{tG}(i,a;j,b)$$

for any $j \in E$ and $b \in F$ where e^{tG} is the matrix exponential

$$e^{tG} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} G^n = \lim_{n\to+\infty}\left(I + \frac{G}{n}\right)^n. \tag{2}$$

Using this fact, we have the explicit representation for system reliability

$$P_{ia}\{L > t\} = \sum_{j\in E}\sum_{b=0}^{M-1} e^{tG}(i,a;j,b). \tag{3}$$

The computation of the matrix exponential (2) can done in many ways and we refer the interested reader to Moler and Loan (2003) for various methods. Most of our results can be stated using the matrix exponential (2) and there are many efficient methods to compute it.

3.2 Mission Reliability

Let $T_0, T_1, T_2, \cdots$ be the transition times of the mission process so that T_n denotes the time at which the nth phase ends. In a given application, it may be important to determine the probability that the system will complete the first n phases successfully, or $P_{ia}\{L > T_n\}$ where $a \neq M$. Note that

$$P_{ia}\{L > T_1\} = \sum_{j\in E}\int_0^{+\infty} P_{ia}\{L > T_1, T_1 \in ds, X_{T_1} = j\} = \sum_{j\in E}\sum_{b=0}^{M-1}\int_0^{+\infty} e^{sG_i}(a,b)\,Q(i,j)\,\mu(i)\,e^{-\mu(i)s}ds.$$

For all $i, j \in E$ and $a, b \in F\backslash\{M\}$, let

$$\begin{aligned} P^*(i,a;j,b) &= \int_0^{+\infty} e^{sG_i}(a,b)\,Q(i,j)\,\mu(i)\,e^{-\mu(i)s}ds = \sum_{n=0}^{+\infty}\frac{G_i^n(a,b)}{n!}Q(i,j)\left(\int_0^{+\infty}\mu(i)\,s^n e^{-\mu(i)s}ds\right) \\ &= \sum_{n=0}^{+\infty}\frac{G_i^n(a,b)}{\mu(i)^n}Q(i,j) = \left(I - \frac{1}{\mu(i)}G_i\right)^{-1}(a,b)\,Q(i,j). \end{aligned} \tag{4}$$

This is a computationally tractable matrix which can found by taking a matrix inverse for every phase.

Then, letting $f_{ia}^{(n)} = P_{ia}\left\{L > T_n\right\}$, we have

$$f_{ia}^{(1)} = P_{ia}\left\{L > T_1\right\} = \sum_{j\in E}\sum_{b=0}^{M-1} P^*\left(i,a;j,b\right) = P^*\overline{1}\left(i,a\right)$$

for $n = 1$. Conditioning on the state after the first transition, we obtain

$$f_{ia}^{(n+1)} = P_{ia}\left\{L > T_{n+1}\right\} = \sum_{j\in E}\sum_{b=0}^{M-1} P^*\left(i,a;j,b\right) P_{jb}\left\{L > T_n\right\}$$

so that

$$f^{(n+1)} = P^* f^{(n)}$$

with the boundary condition $f_{ia}^{(0)} = \overline{1}$. Therefore, the mission reliability for the first n phases is

$$f^{(n)} = P_{ia}\left\{L > T_n\right\} = \left(P^*\right)^n \overline{1}(i,a) \tag{5}$$

which is actually the row sum of the nth power of the matrix P^* on $E \times (F\backslash\{M\})$.

3.3 Phase Reliability

Depending on the overall objective of the mission, a given critical phase may be more important than the others for a complex system. Therefore, an important measure to represent the reliability of the system may be the probability that this critical phase will be completed in a fixed time period. Suppose that one is interested in the successful completion of a given critical phase $j \in E$ of the mission. Letting

$$U_j = \inf\{t \geq 0; X_t \neq X_{t-} = j\}$$

denote the first time that the mission process leaves state j, the phase reliability is $P_{ia}\{U_j \leq t, L > U_j\}$ which is the probability that phase j is successfully completed before time t.

We now consider a new Markov process $Z = \{Z_t : t \geq 0\}$ by stopping the Markov process $(X, A) = \{(X_t, A_t) : t \geq 0\}$ such that if the critical phase is completed without any failure while the age of the unit is a, then the process Z will jump to an absorbing success state (S, a) where S denotes successful completion of phase j. Then, Z is also a Markov process on the extended state space $E \cup \{S\} \times F$ with infinitesimal generator

$$\widetilde{G}_j\left(i,a;k,b\right) = \begin{cases} \mu\left(j\right) & \text{if } i = j, a \neq M, k = S, b = a \\ G_i\left(a,b\right) & \text{if } k = i, a \neq M, b \neq a \\ H\left(i,k\right) & \text{if } i \neq j, a \neq M, k \neq i, b = a \\ G_i\left(a,a\right) + H\left(i,i\right) & \text{if } a \neq M, k = i, b = a \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The reliability of phase j is given by the matrix exponential

$$P_{ia}\{U_j \leq t, L > U_j\} = \sum_{b=0}^{M-1} P_{ia}\left\{Z_t = (S,b)\right\} = \sum_{b=0}^{M-1} e^{t\widetilde{G}_j}\left(i,a;S,b\right). \tag{7}$$

4 Numerical Illustration

Suppose that the mission has 3 phases and the system has 3 deterioration levels so that $E = \{1,2,3\}$ and $F = \{0,1,2\}$ where $M = 2$ denotes failure. Suppose arbitrarily that $\mu = (1, 2, 1.5)$, $\lambda_1 = (0.5, 1)$, $\lambda_2 = (1.5, 2)$, $\lambda_3 = (1, 1.5)$ and

$$Q = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.4 & 0 & 0.6 \\ 0.7 & 0.3 & 0 \end{bmatrix}, \; P_1 = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; P_2 = \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; P_3 = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so that

$$H = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 0.8 & -2 & 1.2 \\ 1.05 & 0.45 & -1.5 \end{bmatrix}, \; G_1 = \begin{bmatrix} -0.5 & 0.35 & 0.15 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -1.5 & 1.2 & 0.3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \; G_3 = \begin{bmatrix} -1 & 0.6 & 0.4 \\ 0 & -1.5 & 1.5 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$G = \begin{array}{c} (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \\ (3,0) \\ (3,1) \\ (3,2) \end{array} \begin{bmatrix} -1.5 & 0.35 & 0.15 & 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 0 & 0 & -3.5 & 1.2 & 0.3 & 1.2 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & -4 & 2 & 0 & 1.2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.05 & 0 & 0 & 0.45 & 0 & 0 & -2.5 & 0.6 & 0.4 \\ 0 & 1.05 & 0 & 0 & 0.45 & 0 & 0 & -3 & 1.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, using (1) and (3), system reliability for 2 time units beginning with phase is 1

$$P_{10}\left\{L > 2\right\} = \sum_{j=1}^{3}\sum_{b=0}^{1} e^{2G}\left(1,0;j,b\right) = 0.1352 + 0.0740 + 0.0352 + 0.0283 + 0.0625 + 0.0424 = 0.3776$$

where the matrix exponential is calculated by MATLAB 7.6.0 (R2008a) which uses the scaling and squaring method employing Padé approximants.

To obtain the mission reliability, we use (4) and (5) to compute the probability that the first phase is completed without failure as

$$f^{(1)} = P_{10}\left\{L > T_1\right\} = P^{*}\bar{1}(1,0) = 0.3333 + 0.0583 + 0.3333 + 0.0583 = 0.7832.$$

Similarly, mission reliability for 2 and 3 phases are $P_{10}\left\{L > T_2\right\} = \left(P^{*}\right)^{2}\bar{1}(1,0) = 0.5459$ and $P_{10}\left\{L > T_3\right\} = \left(P^{*}\right)^{3}\bar{1}(1,0) = 0.3739$ after taking the second and the third powers of P^{*}.

Finally, the probability that the critical phase $j = 2$ is completed until time 1.5 is

$$P_{10}\{U_2 \leq 1.5, L > U_2\} = \sum_{b=0}^{1} e^{1.5\widetilde{G}_j}\left(1,0;S,b\right) = 0.1895 + 0.0754 = 0.2649$$

using (6) and (7).

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