Hitting times of a deterministic or random threshold by a non-stationary gamma process

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Abstract

We consider the gamma process as a degradation model. The hitting time of a deterministic or random level is studied here. In the first case, the distribution is given explicitly. In the second case, we propose to approximate the general case by considering mixtures of Erlang distributions.

1 Introduction

Gamma process is one of the most popular stochastic process to model degradation of device in reliability theory. See the review by van Noortwijk (2009) in the maintenance context. We recall the definition of this random process. Let $\xi \in \mathbb{R}^+$ and $\eta = (\eta_t)_{t \geqslant 0}$ be a real increasing function with $\eta_0 = 0$ and $\lim_{t \to \infty} \eta_t = \infty$. One says that (D_t) is gamma process if and only if it satisfies: (1) $D_0 = 0$; (2) its increments are independents; (3) its increments are gamma distributed. More precisely, for any t and δ, $D_{t+\delta} - D_t$ is a gamma distributed random variable with parameter $(\xi, \eta_{t+\delta} - \eta_t)$. In particular, it implies that all the marginals of such stochastic process are gamma distributed: for any $t \geqslant 0$, the pdf of D_t is:

$$f_{D_t}(x) = \frac{1}{\xi \Gamma(\eta_t)} \left(\frac{x}{\xi} \right)^{\eta_t - 1} e^{-x/\xi} \mathbb{1}_{\mathbb{R}^+}(x) ,$$

where $\Gamma(\cdot)$ is the gamma function. The case where η is linear, say $\eta_t = \alpha t$, is the most widely considered in the literature. Indeed, in such case, (D_t) has stationary increments: for any t and δ, $D_{t+\delta} - D_t$ and D_t are identically distributed. Then (D_t) turns to be a Lévy process. This particular case allows many computations. Some non-linear cases are also studied. Among them, the one where η is a power function, say $\eta_t = \alpha t^\beta$, is the most frequently considered. For such model, β is generally assumed to be known when dealing with parameter estimation. Lawless and Crowder (2004) have considered a different non-linear shape function by considering the Paris-Erdogan curve for η_t in order to model crack propagation.

Here we will consider the general non-stationary case, even the linear case will be studied as a particular case. We will study the hitting distribution of a level. This level will be first assumed to be deterministic and then random.

2 Hitting time distribution

The failure of a system is often modelling throughout a degradation process. In such case, one assume generally that the system will fail as soon as the degradation level is higher than a certain threshold. Two cases will be considered here: the first one is when the threshold is deterministic and the second one is when the threshold is random.

2.1 Case of deterministic threshold

Let c be a positive constant corresponding the critical degradation level leading to the failure of the system. Thus we consider the following random variable T_c defined as follows:

$$T_c = \inf \{ t \geqslant 0 \ ; \ D_t \geqslant c \} \ .$$

Since the stochastic process (D_t) has increasing paths, one gets:

$$\forall t \geqslant 0 \ , \quad \mathbb{P}[T_c > t] = \mathbb{P}[D_t < c] \ .$$

The following theorem generalizes a result proved by Park et Padgett (2005) (they only consider the linear case):

Theorem 1. *The cumulative distribution function of T_c is:*

$$\forall t \geqslant 0 \ , \quad F_{T_c}(t) = \frac{\Gamma(\eta_t, c/\xi)}{\Gamma(\eta_t)} \ ,$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete Gamma function. Moreover, if η is derivable, the probability distribution function of T_c is:

$$\forall t \geqslant 0 \ , \quad f_{T_c}(t) = \eta_t' \left(\Psi(\eta_t) - \log\left(\frac{c}{\xi}\right)\right) \frac{\gamma(\eta_t, c/\xi)}{\Gamma(\eta_t)} + \frac{\eta_t'}{\eta_t^2 \Gamma(\eta_t)} \left(\frac{c}{\xi}\right)^{\eta_t} {}_2F_2(\eta_t, \eta_t; \eta_t + 1, \eta_t + 1; -c/\xi) \ ,$$

where Ψ is the di-gamma function (or logarithmic derivative of the Gamma function), $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma function and ${}_2F_2$ is the generalized hypergeometric function of order $(2,2)$.

Other expressions have been proposed: see the survey by van Noortwijk (2009) and the references therein. As mentioned by Park and Padgett (2005), Birnbaum-Saunders distribution can be used to approximate the above distribution.

2.2 Case of random threshold

Since the paper by Abdel-Hameed (1975), some authors have studied the problem of the hitting time distribution of a random level. From the above expression of the cdf, one can derive easily the cdf of T_C where C is a random variable exponentially distributed with parameter ρ.

Theorem 2. *The cumulative distribution function of T_C is:*

$$\forall t \geqslant 0 \ , \quad F_{T_C}(t) = 1 - (1 + \rho\xi)^{-\eta_t} \ ,$$

and the probability distribution function of T_C is:

$$\forall t \geqslant 0 \ , \quad f_{T_C}(t) = \eta_t'(1 + \rho\xi)^{-\eta_t} \log(1 + \rho\xi) \ .$$

This result is the same than the one obtained by Frenk and Nicolai (2007) by a different technique. One can remark that in such case T_C has an increasing failure rate (IFR). Indeed, from the above results, one can easily gets the hazard rate function of T_C:

$$\forall t \geqslant 0 \ , \quad h_{T_C}(t) = \eta_t' \log(1 + \rho\xi) \ .$$

Moreover the assumption on η implies that η' is a non-decreasing function. It follows that h_{T_C} is also a non-decreasing function and that T_C is IFR.

Computations are not so easily tractable for any distribution of C. Anyway one can approximate this distribution by a phase-type distribution (Neuts 1994):

Theorem 3. *Let C be a positive random variable and C_n be a phase-type distribution such that C_n converges in distribution to C as n tends to infinity. Thus, T_{C_n} converges in distribution to T_C as n tends to infinity.*

For instance, any positive random variable can be approximated by a mixture of Erlang distributions with the same shape parameter. In such case, computations can be easily derived and one can prove the following result:

Theorem 4. *Assume that C_n is a mixture of Erlang distributions as follows:*

$$\forall x \in \mathbb{R}_+ \ , \quad f_{C_n}(x) = \sum_{i=1}^{n} p_i \rho^{k_i} \frac{x^{k_i - 1}}{(k_i - 1)!} e^{-\rho x} \ ,$$

with $(p_1, \ldots, p_n)$ such that $p_1 + \cdots + p_n = 1$, $\rho > 0$ and $k_1, \ldots, k_n > 0$. Then the cdf of T_{C_n} is:

$$\forall t \geqslant 0 \ , \quad F_{T_{C_n}}(t) = \sum_{i=1}^{n} \frac{p_i}{k_i!} (\eta_t)_{k_i} \left(\frac{\rho\xi}{1 + \rho\xi}\right)^{k_i} (1 + \rho\xi)^{-\eta_t} {}_2F_1\left(1, \eta_t + k_i; k_i; \frac{\rho\xi}{1 + \rho\xi}\right) \ .$$

2.3 Stochastic comparison

The hitting times of a threshold by a gamma process preserves the (usual) stochastic order for the threshold. This is trivial in the case of deterministic threshold since a gamma process has increasing paths. We recall the definition of the stochastic order (Stoyan 1983): we said that the random variable X is stochastically smaller than the random variable Y, $X \preceq_{st} Y$, if and only if for any t, $\mathbb{P}[X > t] \leqslant \mathbb{P}[Y > t]$. The following theorem holds:

Theorem 5. *Let C_1 and C_2 be two positive random variables such that $C_1 \preceq_{st} C_2$. Then $T_{C_1} \preceq_{st} T_{C_2}$.*

References

Abdel-Hameed, M. (1975). A gamma wear process. *IEEE Trans. Reliab. 24*(2), 152–153.

Frenk, J. and Nicolai, R. (2007). Approximating the randomized hitting time distribution of a non-stationary gamma process. ERIM Report Series Reference No. ERS-2007-031-LIS.

Lawless, J. and Crowder, M. (2004). Covariates and random effects in a gamma process model with application to degradation and failure. *Lifetime Data Analysis 10*, 213–227.

Neuts, M.F. (1994). *Matrix-geometric solutions in stochastic models: an algorithmic approach.* Dover Publisher, New-York.

Park, C. and Padgett, W. (2005). Accelerated degradation models for failure based on geometric brownian motion and gamma processes. *Lifetime Data Analysis 11*, 511–527.

Stoyan, D. (1983). *Comparison methods for queues and other stochastic models.* Wiley, Chichester.

van Noortwijk, J.M. (2009). A survey of the application of gamma processes in maintenance. *Reliability Engineering and System Safety 94*, 2–21.