Stochastic comparisons of residual lifetimes in multivariate frailty models

Franco Pellerey
Politecnico di Torino
Italy
franco.pellerey@polito.it

Julio Mulero
Universidad de Murcia
Spain
jmulero@um.es

Rosario Rodríguez-Griñolo
Universidad Pablo de Olavide, Sevilla
Spain
mrrodgri@upo.es

Abstract

Consider two vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ of random lifetimes whose distributions are defined via the frailty approach, and let $\mathbf{X}_{k,\mathbf{t}} = [\mathbf{X}_k - \mathbf{t} | \, \mathbf{X}_k > \mathbf{t}]$, $k = 1, 2$, be the corresponding vectors of residual lifetimes at $\mathbf{t} = (t_1, \ldots, t_n), t_i \in \mathbb{R}^+, i = 1, \ldots, n$. Here we describe sufficient conditions for the stochastic comparison $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{2,\mathbf{t}}$ for every vector $\mathbf{t}$ of non-negative times.

1 Introduction

A vector $\mathbf{X}_k = (X_{k,1}, \ldots, X_{k,n})$ of non independent lifetimes is said to be described by a frailty model if its joint survival function is defined as

$$\overline{F}_k(t_1, \ldots, t_n) = \mathbb{P}[X_{k,1} > t_1, \ldots, X_{k,n} > t_n] = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^{\Theta_k}\right], \tag{1}$$

where Θ_k is an environmental random frailty taking values in $\mathbb{R}^+$ and $\overline{G}_{k,i}$ is the survival function of lifetime $X_{k,i}$ given $\Theta_k = 1$.

Interesting conditions for the stochastic comparison in different manners between to vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ defined as above have been recently shown in Misra et al. (2009). In particular, in Misra et al. (2009) it is shown that $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$ whenever $\overline{G}_{1,i} = \overline{G}_{2,i}$ for all $i = 1, \ldots, n$ and $\Theta_1 \leq_{st} \Theta_2$, where $\leq_{st}$ is the usual stochastic order.

Here we provide an alternative sufficient condition for $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$, and we describe two of its consequences in comparisons of corresponding vectors of residual lifetimes at any time $\mathbf{t}$. In particular, we show that the inequality $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$ follows also from a different stochastic inequality between the random frailties Θ_1 and Θ_2, called here $\leq_{Lt-lr}$, whose definition is the following.

Definition 1. Given to non-negative random variables Θ_1 and Θ_2 we say that Θ_1 is smaller than Θ_2 in the Laplace transform – likelihood ratio order (shortly $\Theta_1 \leq_{Lt-lr} \Theta_2$) iff the ratio

$$\frac{\mathbf{E}[\Theta_1 \exp(-s\Theta_1)]}{\mathbf{E}[\Theta_2 \exp(-s\Theta_2)]}$$

is decreasing in $s \in \mathbb{R}^+$.

In Section 3 some of its relationships with other well-known univariate stochastic orders will be mentioned; here just observe that the $\leq_{Lt-lr}$ order does not implies, nor is implied by, the $\leq_{st}$ order, and that $\Theta_1 \leq_{Lt-lr} \Theta_2$ holds iff the ratio $\frac{w_1(s)}{w_2(s)}$ is decreasing in s, where

$$w_k(s) = \frac{dW_k(s)}{ds} = \frac{d[1 - \int_0^\infty \exp(-su) dH_k(u)]}{ds}, \tag{2}$$

and where H_k is the cumulative distribution of Θ_k, $k = 1, 2$. In other words, $\Theta_1 \leq_{Lt-lr} \Theta_2$ corresponds to the likelihood ratio order ($\leq_{lr}$) between the variables $\widetilde{\Theta}_k$ having the Laplace transforms of H_k, $k = 1, 2$, as their survival functions.

Two preliminary results are needed for the prosecution. The proof of the first one easily follows from the closure property of log-convexity with respect to mixture and observing that the function $\exp(-su)$ is log-convex in s (see Barlow and Proschan (1975), pag 102, for details).

Lemma 1. *Whatever the distribution of Θ_k is, the corresponding density w_k defined in (2) is log-convex.*

The second preliminary result is stated as Theorem 6.B.4 in Shaked and Shanthikumar (2007). See there for definition of CIS property and for definitions of the well-know stochastic order considered throughout this note.

Lemma 2. *Let $\mathbf{Y}_1 = (Y_{1,1}, \dots, Y_{1,n})$ and $\mathbf{Y}_2 = (Y_{2,1}, \dots, Y_{2,n})$ be two random vectors such that $\mathbf{Y}_1$, or $\mathbf{Y}_2$, is conditionally increasing in sequence (shortly, CIS). Then $\mathbf{Y}_1 \leq_{st} \mathbf{Y}_2$ holds if:*
(i) $Y_{1,1} \leq_{st} Y_{2,1}$;
(ii) $[Y_{1,i}|\ Y_{1,1} = t_1, \dots, Y_{1,i-1} = t_{i-1}] \leq_{st} [Y_{2,i}|\ Y_{2,1} = t_1, \dots, Y_{2,i-1} = t_{i-1}]$ $\forall i = 2, \dots, n$ and $t_j \geq 0$, with $j = 1, \dots, i-1$.

2 Main results

The first result describes conditions for the usual stochastic comparison between two frailty models.

Theorem 1. *Let the vectors $\mathbf{X}_k$, with $k = 1, 2$, have survival functions defined as in (1). If:*
(a) $\Theta_1 \leq_{Lt-lr} \Theta_2$;
(b) $[X_{1,i}|\Theta_1 = 1] \leq_{st} [X_{2,i}|\Theta_2 = 1]$ $\forall i = 1, \dots, n$,
then $\mathbf{X}_1 \leq_{st} \mathbf{X}_2$.

Proof. Let us consider a vector $\mathbf{Y}$ having joint survival function $\overline{F}_{\mathbf{Y}}(t_1, \dots, t_n) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{2,i}(t_i)\right)^{\Theta_1}\right]$. First we will see that $\mathbf{Y} \leq_{st} \mathbf{X}_2$.

For it, let us observe that the vector $\mathbf{X}_2$ satisfies the CIS property, as it can be proved directly with some calculations or by using Lemma 1 and Proposition 1 in Averous and Dortet–Bernadet (2004). Thus, to prove the assertion it suffices to verity that assumptions (i) and (ii) in Lemma 2 are satisfied.

Note that, for all $t_1 \in \mathbb{R}^+$,

$$\begin{aligned}\overline{F}_{\mathbf{Y},1}(t_1) = \mathbf{E}[\overline{G}_{2,1}(t_1)^{\Theta_1}] &= \mathbf{E}[\exp(-\Theta_1 \ln \overline{G}_{2,1}(t_1))] \\ &\leq \mathbf{E}[\exp(-\Theta_2 \ln \overline{G}_{2,1}(t_1))] = \mathbf{E}[\overline{G}_{1,1}(t_1)^{\Theta_2}] = \overline{F}_{2,1}(t_1),\end{aligned}$$

where the inequality follows from assumption (a). Thus (i) in Lemma 2 holds.

Moreover, for all $i = 1, \dots, n$ and $t_j \geq 0, j = 1, \dots, i$, it holds

$$\begin{aligned}\overline{F}_{\mathbf{Y},i|Y_1=t_1,\dots,Y_{i-1}=t_{i-1}}(t_i) &= \frac{w_1(-\ln \overline{G}_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))}{w_1(-\sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))} \\ &\leq \frac{w_2(-\ln \overline{G}_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))}{w_2(-\sum_{j=1}^{i-1} \ln \overline{G}_{2,j}(t_j))} = \overline{F}_{2,i|X_{2,1}=t_1,\dots,X_{2,i-1}=t_{i-1}}(t_i),\end{aligned}$$

where, again, the inequality follows from assumption (a). Thus, also assumption (ii) in Lemma 2 is satisfied. We can then assert that $\mathbf{Y} \leq_{st} \mathbf{X}_2$.

Now observe that, by Theorem 6.B.14 in Shaked and Shanthikumar (2007), it holds $\mathbf{X}_1 \leq_{st} \mathbf{Y}$, having the vectors $\mathbf{X}_1$ and $\mathbf{Y}$ the same copula and stochastically ordered margins (by assertion (b) and closure of $\leq_{st}$ with respect to mixture).

The main assertion now follows from $\mathbf{X}_1 \leq_{st} \mathbf{Y} \leq_{st} \mathbf{X}_2$. □

Under a stronger assumption (b) it is possible to get a stronger comparison between $\mathbf{X}_1$ and $\mathbf{X}_2$, which involves the vectors of their residual lifetimes.

Theorem 2. *Let the vectors $\mathbf{X}_k$, with $k = 1, 2$, have survival functions defined as in (1). If:*
(a) $\Theta_1 \leq_{Lt-lr} \Theta_2$;
(b) $[X_{1,i}|\Theta_1 = 1] \leq_{hr} [X_{2,i}|\Theta_2 = 1]$ $\forall i = 1, \dots, n$;
then $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{2,\mathbf{t}}$ for every vector $\mathbf{t} = (t_1, \dots, t_n)$ of non-negative real values.

Proof. Let $\mathbf{u} = (u_1, \ldots, u_n)$ be any vector of non-negative values. Note that

$$\begin{aligned}
\overline{F}_{\mathbf{X}_{k,\mathbf{t}}}(\mathbf{u}) = \frac{\overline{F}_k(\mathbf{t}+\mathbf{u})}{\overline{F}_k(\mathbf{t})} &= \frac{\int_0^\infty \left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i+u_i)\right)^\theta dH_k(\theta)}{\int_0^\infty \left(\Pi_{i=1}^n \overline{G}_{k,i}(t_i)\right)^\theta dH_k(\theta)} \\
&= \frac{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j+u_j)]\} dH_k(\theta)}{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)} \\
&= \int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln(\frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)})]\} \frac{\exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)}{\int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)} \\
&= \int_0^\infty \exp\{\theta[\sum_{j=1}^n \ln(\frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)})]\} d\widetilde{H}_k(\theta).
\end{aligned}$$

Thus, $\mathbf{X}_{k,\mathbf{t}}$ has joint survival function which can be expressed as

$$\overline{F}_{\mathbf{X}_{k,\mathbf{t}}}(\mathbf{u}) = \mathbf{E}\left[\left(\Pi_{i=1}^n \overline{G}_{k,i,t_i}(u_i)\right)^{\widetilde{\Theta}_k}\right]$$

where

$$G_{k,i,t_i}(u_i) = \frac{\overline{G}_{k,j}(t_j+u_j)}{\overline{G}_{k,j}(t_j)}$$

and where $\widetilde{\Theta}_k$ has distribution $\widetilde{H}_k$ defined as

$$\widetilde{H}_k(\theta) = \frac{\int_0^\theta \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\tau)}{\int_0^\infty \exp\{\tau[\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\tau)} .$$

Thus, also,

$$\mathbf{E}[\exp(-s\widetilde{\Theta}_k)] = \frac{\mathbf{E}[\exp(-(s+\widetilde{t}_k)\Theta_k)]}{\mathbf{E}[\exp(-\widetilde{t}_k\Theta_k)]},$$

where $\widetilde{t}_k = -\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)$.

Let us denote with

$$\widetilde{w}_{k,\mathbf{t}}(s) = \frac{d\mathbf{E}[\exp(-s\widetilde{\Theta}_k)]}{ds},$$

the derivative of the Laplace transform of $\widetilde{H}_k$.

It holds

$$\frac{\widetilde{w}_{1,\mathbf{t}}(s)}{\widetilde{w}_{2,\mathbf{t}}(s)} = \frac{\mathbf{E}[\exp(-\widetilde{t}_2\Theta_2)]}{\mathbf{E}[\exp(-\widetilde{t}_1\Theta_1)]} \cdot \frac{w_{1,\mathbf{t}}(s+\widetilde{t}_1)}{w_{2,\mathbf{t}}(s+\widetilde{t}_2)} = \frac{\mathbf{E}[\exp(-\widetilde{t}_2\Theta_2)]}{\mathbf{E}[\exp(-\widetilde{t}_1\Theta_1)]} \cdot \frac{w_{1,\mathbf{t}}(s+\widetilde{t}_2)}{w_{2,\mathbf{t}}(s+\widetilde{t}_2)} \cdot \frac{w_{1,\mathbf{t}}(s+\widetilde{t}_1)}{w_{1,\mathbf{t}}(s+\widetilde{t}_2)}.$$

Since $\frac{w_{1,\mathbf{t}}(s+\widetilde{t}_2)}{w_{2,\mathbf{t}}(s+\widetilde{t}_2)}$ is decreasing in s by assumption (a), while $\frac{w_{1,\mathbf{t}}(s+\widetilde{t}_1)}{w_{1,\mathbf{t}}(s+\widetilde{t}_2)}$ is decreasing in s because of Lemma 1 and $\widetilde{t}_1 \geq \widetilde{t}_2$, as one can verify, it holds $\widehat{\Theta}_1 \leq_{Lt-lr} \widehat{\Theta}_2$.

Moreover, from assumption (b) easily follows that $[X_{1,i,t_i}|\widehat{\Theta}_1 = 1] \leq_{st} [X_{2,i,t_i}|\widehat{\Theta}_2 = 1]$ $\forall i = 1, \ldots, n$.

Thus one can apply Theorem 1 to $\mathbf{X}_{1,\mathbf{t}}$ and $\mathbf{X}_{2,\mathbf{t}}$, getting the assertion. □

Using arguments similar to those in the proof of Theorem 2 it is possible to prove also the following result, which describes conditions for a particular notion of negative multivariate aging.

Theorem 3. *Let the vector* $\mathbf{X}_1$ *have survival function defined as in (1). Then* $\mathbf{X}_{1,\mathbf{t}} \leq_{st} \mathbf{X}_{1,\mathbf{t}+\mathbf{u}}$ *holds for every* $\mathbf{t} = (t_1, \ldots, t_n)$ *and* $\mathbf{u} = (u_1, \ldots, u_n)$ *if for all* $i = 1, \ldots, n$ *the variable* $[X_{1,i}|\ \Theta_1 = 1]$ *has decreasing hazard rate, i.e.,* $[X_{1,i,t_i}|\ \Theta_1 = 1] \leq_{st} [X_{1,i,t_i+u_i}|\Theta_1 = 1]$ $\forall t_i, u_i \geq 0$.

This result is not surprising, in particular if compared with similar conditions for other notions of negative multivariate aging (see, e.g., Spizzichino and Torrisi, 2001).

Remark 1. Other results of this kind, bur for weaker multivariate stochastic orders, may be found in Mulero et al. (2008).

3 The Laplace transform – likelihood ratio order

The $\leq_{Lt-lr}$ has been never considered before in the literature. However, it is strictly related to the orders $\leq_{Lt-r}$ and $\leq_{r-Lt-r}$, which are based on monotonicity properties of ratios of Laplace transforms and defined and studied in Shaked and Wong (1997). In fact, using standard TP_2 techniques, it can be proved that $\Theta_1 \leq_{Lt-lr} \Theta_2$ implies both $\Theta_1 \leq_{Lt-r} \Theta_2$ and $\Theta_1 \leq_{r-Lt-r} \Theta_2$.

Moreover, like the $\leq_{Lt-r}$ and $\leq_{r-Lt-r}$ orders, it does not imply the usual stochastic order $\leq_{st}$. To prove it, it suffices to consider the variables Θ_1 and Θ_2 having discrete densities f_{Θ_k} defined as

$$f_{\Theta_1}(t) = \begin{cases} 0.2 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.4 & \text{if } t = 2.9 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{\Theta_2}(t) = \begin{cases} 0.3 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.3 & \text{if } t = 3 \\ 0 & \text{otherwise.} \end{cases}$$

With some straightforward calculation it is easy to verify that $\frac{w_1(s)}{w_2(s)}$ is decreasing in s, i.e., $\Theta_1 \leq_{Lt-lr} \Theta_2$, while $\Theta_1 \leq_{st} \Theta_2$ is not satisfied since their survivals do intersect. Moreover, the usual stochastic order does not imply the Laplace transform – likelihood ratio order, since it does not imply the $\leq_{Lt-r}$ and $\leq_{r-Lt-r}$ orders (see Shaked and Wong, 1997).

References

[1] Avérous, J. and Dortet–Bernadet, M.B. (2004). Dependence for Archimedean copulas and aging properties of their generating functions. *Sankhya*, **66**, 607–620.

[2] Barlow, R.E. and Proshan, F. (1975), *Statistical Theory of Reliability and Life Testing: Probability Models,* Hold, Rinehart and Winston. NewYork.

[3] Misra, N., Gupta, N. and Gupta,R.D. (2009). Stochastic comparisons of multivariate frailty models. To appear in *Journal of Statistical Planning and Inference, doi: 10.1016/j.jspi.2008.09.006*.

[4] Mulero, J., Pellerey, F. and Rodríguez-Griñolo, R. (2008). Stochastic comparisons for time transformed exponential models. *Technical Report n. 13–2008, Dipartimento di Matematica, Politecnico di Torino, Torino, Italy.*

[5] Shaked, M. and Shanthikumar, J.G. (2007), *Stochastic orders*, Springer Verlag, New York.

[6] Shaked, M. and Wong, T. (1997). Stochastic orders based on ratios of Laplace transforms. *Journal of Applied Probability* **34**, 404–419.

[7] Spizzichino, F. and Torrisi, G.L. (2001). Multivariate negative aging in an exchangeable model of heterogeneity. *Statistics and Probability Letters* **55**, 71–82.