Abstract

Consider two vectors $X_1$ and $X_2$ of random lifetimes whose distributions are defined via the frailty approach, and let $X_{k,t} = [X_k - t | X_k > t]$, $k = 1, 2$, be the corresponding vectors of residual lifetimes at $t = (t_1, \ldots, t_n), t_i \in \mathbb{R}^+, i = 1, \ldots, n$. Here we describe sufficient conditions for the stochastic comparison $X_{1,t} \preceq_{st} X_{2,t}$ for every vector $t$ of non-negative times.

1 Introduction

A vector $X_k = (X_{k,1}, \ldots, X_{k,n})$ of non independent lifetimes is said to be described by a frailty model if its joint survival function is defined as

$$F_k(t_1, \ldots, t_n) = \mathbb{P}[X_{k,1} > t_1, \ldots, X_{k,n} > t_n] = \mathbb{E} \left[ \left( \prod_{i=1}^{n} G_{k,i}(t_i) \right)^{\Theta_k} \right], \quad (1)$$

where $\Theta_k$ is an environmental random frailty taking values in $\mathbb{R}^+$ and $G_{k,i}$ is the survival function of lifetime $X_{k,i}$ given $\Theta_k = 1$.

Interesting conditions for the stochastic comparison in different manners between to vectors $X_1$ and $X_2$ defined as above have been recently shown in Misra et al. (2009). In particular, in Misra et al. (2009) it is shown that $X_1 \preceq_{st} X_2$ whenever $G_{1,i} = G_{2,i}$ for all $i = 1, \ldots, n$ and $\Theta_1 \leq_{lr} \Theta_2$, where $\leq_{st}$ is the usual stochastic order.

Here we provide an alternative sufficient condition for $X_1 \preceq_{st} X_2$, and we describe two of its consequences in comparisons of corresponding vectors of residual lifetimes at any time $t$. In particular, we show that the inequality $X_1 \preceq_{st} X_2$ follows also from a different stochastic inequality between the random frailties $\Theta_1$ and $\Theta_2$, called here $\leq_{Lt-lr}$, whose definition is the following.

**Definition 1.** Given to non-negative random variables $\Theta_1$ and $\Theta_2$ we say that $\Theta_1$ is smaller than $\Theta_2$ in the Laplace transform – likelihood ratio order (shortly $\Theta_1 \leq_{Lt-lr} \Theta_2$) iff the ratio

$$\frac{\mathbb{E}[\Theta_1 \exp(-s\Theta_1)]}{\mathbb{E}[\Theta_2 \exp(-s\Theta_2)]}$$

is decreasing in $s \in \mathbb{R}^+$.

In Section 3 some of its relationships with other well-known univariate stochastic orders will be mentioned; here just observe that the $\leq_{Lt-lr}$ order does not implies, nor is implied by, the $\leq_{st}$ order, and that $\Theta_1 \leq_{Lt-lr} \Theta_2$ holds iff the ratio $\frac{w_1(s)}{w_2(s)}$ is decreasing in $s$, where

$$w_k(s) = \frac{dW_k(s)}{ds} = \frac{d[1 - \int_0^\infty \exp(-su) dH_k(u)]}{ds}, \quad (2)$$

and where $H_k$ is the cumulative distribution of $\Theta_k$, $k = 1, 2$. In other words, $\Theta_1 \leq_{Lt-lr} \Theta_2$ corresponds to the likelihood ratio order ($\leq_{lr}$) between the variables $\Theta_k$ having the Laplace transforms of $H_k$, $k = 1, 2$, as their survival functions.

Two preliminary results are needed for the prosecution. The proof of the first one easily follows from the closure property of log-convexity with respect to mixture and observing that the function $\exp(-su)$ is log-convex in $s$ (see Barlow and Proschan (1975), pag 102, for details).
Lemma 1. Whatever the distribution of $\Theta_k$ is, the corresponding density $w_k$ defined in (2) is log-convex.

The second preliminary result is stated as Theorem 6.B.4 in Shaked and Shanthikumar (2007). See there for definition of CIS property and for definitions of the well-known stochastic order considered throughout this note.

Lemma 2. Let $Y_1 = (Y_{1,1}, \ldots, Y_{1,n})$ and $Y_2 = (Y_{2,1}, \ldots, Y_{2,n})$ be two random vectors such that $Y_1$, or $Y_2$, is conditionally increasing in sequence (shortly, CIS). Then $Y_1 \leq_{st} Y_2$ holds if:

(i) $Y_{1,1} \leq_{st} Y_{2,1}$;

(ii) $|Y_{1,i}| Y_{1,1} = t_1, \ldots, Y_{1,i-1} = t_{i-1}] \leq_{st} |Y_{2,i}| Y_{2,1} = t_1, \ldots, Y_{2,i-1} = t_{i-1}] \forall i = 2, \ldots, n$ and $t_j \geq 0$, with $j = 1, \ldots, i - 1$.

2 Main results

The first result describes conditions for the usual stochastic comparison between two frailty models.

Theorem 1. Let the vectors $X_k$, with $k = 1, 2$, have survival functions defined as in (1). If:

(a) $\Theta_1 \leq_{LT-IR} \Theta_2$;

(b) $[X_{1,i}]\Theta_1 = 1 \leq_{st} [X_{2,i}]\Theta_2 = 1 \forall i = 1, \ldots, n$,

then $X_1 \leq_{st} X_2$.

Proof. Let us consider a vector $Y$ having joint survival function $F_Y(t_1, \ldots, t_n) = E\left[\left(\prod_{i=1}^{n} \Theta_i(t_i)\right)^{\Theta_k}\right]$. First we will see that $Y \leq_{st} X_2$.

For it, let us observe that the vector $X_2$ satisfies the CIS property, as it can be proved directly with some calculations or by using Lemma 1 and Proposition 1 in Averous and Dortet–Bernadet (2004). Thus, to prove the assertion it suffices to verify that assumptions (i) and (ii) in Lemma 2 are satisfied.

Note that, for all $t_1 \in \mathbb{R}^+$,

$$F_{Y,1}(t_1) = E[\Theta_{1,1}(t_1)^{\Theta_1}] = E[\exp(-\Theta_1 \ln \Theta_{1,1}(t_1))] \leq E[\exp(-\Theta_2 \ln \Theta_{1,1}(t_1))] = E[\Theta_{1,1}(t_1)^{\Theta_2}] = F_{2,1}(t_1),$$

where the inequality follows from assumption (a). Thus (i) in Lemma 2 holds.

Moreover, for all $i = 1, \ldots, n$ and $t_j \geq 0$, $j = 1, \ldots, i$, it holds

$$F_{Y,1}[Y_{1,1} = t_1, \ldots, Y_{1,i-1} = t_{i-1}] = \frac{w_1(-\ln \Theta_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \Theta_{2,j}(t_j))}{w_2(-\sum_{j=1}^{i-1} \ln \Theta_{2,j}(t_j))} \leq \frac{w_2(-\ln \Theta_{2,i}(t_i) - \sum_{j=1}^{i-1} \ln \Theta_{2,j}(t_j))}{w_2(-\sum_{j=1}^{i-1} \ln \Theta_{2,j}(t_j))} = F_{2,1}[X_{2,1} = t_1, \ldots, X_{2,i-1} = t_{i-1}],$$

where, again, the inequality follows from assumption (a). Thus, also assumption (ii) in Lemma 2 is satisfied. We can then assert that $Y \leq_{st} X_2$.

Now observe that, by Theorem 6.B.14 in Shaked and Shanthikumar (2007), it holds $X_1 \leq_{st} Y$, having the vectors $X_1$ and $Y$ the same copula and stochastically ordered margins (by assertion (b) and closure of $\leq_{st}$ with respect to mixture).

The main assertion now follows from $X_1 \leq_{st} Y \leq_{st} X_2$.

Under a stronger assumption (b) it is possible to get a stronger comparison between $X_1$ and $X_2$, which involves the vectors of their residual lifetimes.

Theorem 2. Let the vectors $X_k$, with $k = 1, 2$, have survival functions defined as in (1). If:

(a) $\Theta_1 \leq_{LT-IR} \Theta_2$;

(b) $[X_{1,i}]\Theta_1 = 1 \leq_{hr} [X_{2,i}]\Theta_2 = 1 \forall i = 1, \ldots, n$;

then $X_{1,t} \leq_{st} X_{2,t}$ for every vector $t = (t_1, \ldots, t_n)$ of non-negative real values.
Proof. Let \( u = (u_1, \ldots, u_n) \) be any vector of non-negative values. Note that

\[
F_{X_{k,t}}(u) = \frac{F_k(t+u)}{F_k(t)} = \frac{\int_0^\infty \left( \prod_{i=1}^n \overline{G}_{k,i}(t_i + u_i) \right)^\theta dH_k(\theta)}{\int_0^\infty \left( \prod_{i=1}^n \overline{G}_{k,i}(t_i) \right)^\theta dH_k(\theta)} = \frac{\int_0^\infty \exp\{\theta [\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j + u_j)]\} dH_k(\theta)}{\int_0^\infty \exp\{\theta [\sum_{j=1}^n \ln \overline{G}_{k,j}(t_j)]\} dH_k(\theta)} = \frac{\int_0^\infty \exp\{\theta [\sum_{j=1}^n \ln \overline{G}_{k,j,j}(t_j + u_j)]\} \exp\{\theta \sum_{j=1}^n \ln \overline{G}_{k,j,j}(t_j)\} dH_k(\theta)}{\int_0^\infty \exp\{\theta [\sum_{j=1}^n \ln \overline{G}_{k,j,j}(t_j)]\} dH_k(\theta)}
\]

Thus, \( X_{k,t} \) has joint survival function which can be expressed as

\[
F_{X_{k,t}}(u) = E\left[ \left( \prod_{i=1}^n \overline{G}_{k,i,i}(u_i) \right)^\tilde{\Theta}_k \right]
\]

where

\[
\overline{G}_{k,i,i}(u_i) = \frac{\overline{G}_{k,j,j}(t_j + u_j)}{G_{k,j,j}(t_j)}
\]

and where \( \tilde{\Theta}_k \) has distribution \( \tilde{H}_k \) defined as

\[
\tilde{H}_k(\tau) = \frac{\int_0^\tau \exp\{\theta [\sum_{j=1}^n \ln \overline{G}_{k,j,j}(t_j)]\} dH_k(\tau)}{\int_0^\infty \exp\{\theta [\sum_{j=1}^n \ln \overline{G}_{k,j,j}(t_j)]\} dH_k(\tau)}.
\]

Thus, also,

\[
E[\exp(-s \tilde{\Theta}_k)] = \frac{E[\exp(-s + \tilde{t}_k) \Theta_k)]}{E[\exp(-\tilde{t}_k \Theta_k)]},
\]

where \( \tilde{t}_k = -\sum_{j=1}^n \ln \overline{G}_{k,j,j}(t_j) \).

Let us denote with

\[
\tilde{w}_{k,t}(s) = \frac{dE[\exp(-s \tilde{\Theta}_k)]}{ds},
\]

the derivative of the Laplace transform of \( \tilde{H}_k \).

It holds

\[
\frac{\tilde{w}_{1,t}(s)}{\tilde{w}_{2,t}(s)} = \frac{E[\exp(-\tilde{t}_2 \Theta_2)]}{E[\exp(-\tilde{t}_1 \Theta_1)]} \cdot \frac{w_{1,t}(s + \tilde{t}_1)}{w_{2,t}(s + \tilde{t}_2)} = \frac{E[\exp(-\tilde{t}_2 \Theta_2)]}{E[\exp(-\tilde{t}_1 \Theta_1)]} \cdot \frac{w_{1,t}(s + \tilde{t}_1)}{w_{2,t}(s + \tilde{t}_2)} \cdot \frac{w_{1,t}(s + \tilde{t}_2)}{w_{1,t}(s + \tilde{t}_1)},
\]

Since \( \frac{w_{1,t}(s + \tilde{t}_1)}{w_{1,t}(s + \tilde{t}_2)} \) is decreasing in \( s \) by assumption (a), while \( \frac{w_{1,t}(s + \tilde{t}_1)}{w_{1,t}(s + \tilde{t}_2)} \) is decreasing in \( s \) because of Lemma 1 and \( \tilde{t}_1 \geq \tilde{t}_2 \), as one can verify, it holds \( \tilde{\Theta}_1 \leq \tilde{\Theta}_2 \).

Moreover, from assumption (b) easily follows that \( [X_{1,i,t_i} | \tilde{\Theta}_1 = 1] \leq_{st} [X_{2,i,t_i} | \tilde{\Theta}_2 = 1] \forall i = 1, \ldots, n. \)

Thus one can apply Theorem 1 to \( X_{1,t} \) and \( X_{2,t} \), getting the assertion.

Using arguments similar to those in the proof of Theorem 2 it is possible to prove also the following result, which describes conditions for a particular notion of negative multivariate aging.

**Theorem 3.** Let the vector \( X \) have survival function defined as in (1). Then \( X_{1,t} \leq_{st} X_{1,t+u} \) holds for every \( t = (t_1, \ldots, t_n) \) and \( u = (u_1, \ldots, u_n) \) if for all \( i = 1, \ldots, n \) the variable \( [X_{1,i} | \Theta_1 = 1] \) has decreasing hazard rate, i.e., \( [X_{1,i,t_i} | \Theta_1 = 1] \leq_{st} [X_{1,i,t_i+u_i} | \Theta_1 = 1] \forall i, u_i \geq 0. \)

This result is not surprising, in particular if compared with similar conditions for other notions of negative multivariate aging (see, e.g., Spizzichino and Torrisi, 2001).

**Remark 1.** Other results of this kind, but for weaker multivariate stochastic orders, may be found in Mulero et al. (2008).
3 The Laplace transform – likelihood ratio order

The $\leq_{Lt-lr}$ has been never considered before in the literature. However, it is strictly related to the orders $\leq_{Lt-r}$ and $\leq_{r-Lt-r}$, which are based on monotonicity properties of ratios of Laplace transforms and defined and studied in Shaked and Wong (1997). In fact, using standard TP$_2$ techniques, it can be proved that $\Theta_1 \leq_{Lt-lr} \Theta_2$ implies both $\Theta_1 \leq_{Lt-r} \Theta_2$ and $\Theta_1 \leq_{r-Lt-r} \Theta_2$.

Moreover, like the $\leq_{Lt-r}$ and $\leq_{r-Lt-r}$ orders, it does not imply the usual stochastic order $\leq_{st}$. To prove it, it suffices to consider the variables $\Theta_1$ and $\Theta_2$ having discrete densities $f_{\Theta_k}$ defined as

$$f_{\Theta_1}(t) = \begin{cases} 0.2 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.4 & \text{if } t = 2.9 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_{\Theta_2}(t) = \begin{cases} 0.3 & \text{if } t = 1 \\ 0.4 & \text{if } t = 2 \\ 0.3 & \text{if } t = 3 \\ 0 & \text{otherwise} \end{cases}$$

With some straightforward calculation it is easy to verify that $w_1(s)/w_2(s)$ is decreasing in $s$, i.e., $\Theta_1 \leq_{Lt-lr} \Theta_2$, while $\Theta_1 \leq_{st} \Theta_2$ is not satisfied since their survivals do intersect. Moreover, the usual stochastic order does not imply the Laplace transform – likelihood ratio order, since it does not imply the $\leq_{Lt-r}$ and $\leq_{r-Lt-r}$ orders (see Shaked and Wong, 1997).

References


