Exact solutions for the two-terminal reliability of recursive structures: a few directions

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Abstract

We present a few results on the determination of the two-terminal reliability for recursive families of graphs. We first show that it is also possible to find the exact expression of the two-terminal reliability for a new, potentially arbitrarily wide network. Exact solutions are therefore not limited to configurations with a finite tree-width. We also show that the structure of the complex zeros of the two-terminal reliability polynomial is not connected. Finally, we show how the conjunction of the knowledge of the actual form of the solution and the use of recent algorithms should provide further analytical results for increasingly complex configurations of identical elements.

1 Presentation

For more than five decades, network reliability has been a field devoted to the calculation of the connection probability between different sites of a network constituted by edges (links, bonds) and nodes (vertices, sites), each of them having a probability of operating correctly (the reliability). This field, although mainly developed in an applied background has strong relations with graph theory (Biggs 1993), percolation theory (Grimmett 1999), as well as in statistical physics (Shrock 2000; Chang and Shrock 2003). The two-terminal reliability $\mathrm{Rel}_2(s \to t)$, the probability that a source s and a destination t are connected, is known in percolation theory as the connectivity function or pair connectedness.

In the general case, the calculation of the network reliability is very complex when the size of the network increases (of course, series and parallel configuration are trivial).

In a recent work (Tanguy 2007a), we have shown that for networks represented by undirected, recursive families of graphs G, the two-terminal reliability may be expressed as a product of transfer matrices, where individual edge and node reliabilities are exactly taken into account. Such a factorization, already observed for graph coloring polynomials, 2D-percolation in square strips or all-terminal reliability polynomials, originates with the underlying algebraic structure of the graph. These results agree with the assessment of (Wolle 2003), namely that the computation time necessary to solve the problem is polynomial in the "tree-width" of the graph. When all the edges and/or nodes of the graphs are identical (with the same reliability p and ρ, respectively), we get a reliability polynomial $R(p, \rho)$. As with all the other graph polynomials, it may be worthwhile to study the location of their zeros in the complex plane as a function of p, for instance.

We were also able to calculate the exact two- and all-terminal reliability for a simple configuration (a "double fan") for which the tree-width may be arbitrarily large (Tanguy 2007b). The unreliability could be expressed as the product of transfer matrices, in which each individual reliability was included.

The questions we asked ourselves were: is this result an exception ? Is it possible to find a simple solution to a potentially very large graph ? If so, does the structure of the complex zeros when $n \to \infty$ remain connected for all values of ρ ?

2 A new wide parallel architecture

We have displayed on Figure 1 the "parallel" architecture studied in this work. We set that there are many finite paths from the source R to the destination U. It looks therefore as a parallel configuration, with added meshed features.

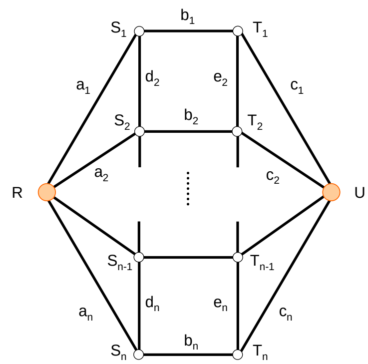

Figure 1: Description of the network architecture. n may be arbitrarily large.

Using techniques developed in (Tanguy 2007a), it has been possible to show that the two-terminal reliability $\mathcal{R}_n$

$$\mathcal{R}_n = RU \left(1 - (1\,0\,0\,0\,0\,0\,0\,0\,0)\; M_n\; M_{n-1}\; \cdots\; M_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

where M is a 9×9 transfer matrix containing all the information for each node or edge of the graph. The form $1 - \cdots$ reminds us of the reliability of a true parallel system.

2.1 Generating function for identical edges and nodes

When edges and nodes have the same reliabilities p and ρ respectively, there is a unique transfer matrix, the elements of which are polynomials in p and ρ. The characteristic polynomial of this transfer matrix is found to be equal to 9, and can be further factorized in polynomials of degrees 3 and 6. We thus expect a recursion relation of order 10 (at most) between successive reliabilities. We can easily calculate their generating function (Stanley 1997); it turns out that the polynomial of degree 3 vanishes. After factorization and simplification, the final result is

$$\mathcal{G}(z) = \frac{p^3\, \rho^4\, z}{1 - z}\; \frac{\mathcal{N}_1(z)}{\mathcal{D}_1(z)}$$

with

$$
\begin{aligned}
\mathcal{N}_1(z) &= 1 + (1-p)\,p\,z\,\rho\left(2 - p\,(1-p+p^2)\,\rho\right) - (1-p)\,p^2\,z^2\,\rho^2\left(1 + p + p\,(6 - 19\,p + 12\,p^2 - 2\,p^3)\,\rho + (3-2\,p)\,p^3\,\rho^2\right) \\
&\quad + (1-p)^3\,p^4\,z^3\,\rho^4\left(1 - p - p^2 + p\,(6 - 13\,p + 13\,p^2 - 4\,p^3)\,\rho - p^3\,(6 - 10\,p + 7\,p^2 - 2\,p^3)\,\rho^2\right) \\
&\quad + (1-p)^4\,p^6\,z^4\,\rho^6\left(1 - 3\,p - p\,(2 - 11\,p + 8\,p^2 - 2\,p^3)\,\rho + p^2\,(1 - 7\,p + 7\,p^2 - 2\,p^3)\,\rho^2\right) \\
&\quad - (1-p)^6\,p^8\,z^5\,\rho^8\left(1 - 3\,p + 3\,p^2 - p^2\,(1+p)\,\rho + p^2\,\rho^2\right) \\
\mathcal{D}_1(z) &= 1 + z\left(-1 - 2\,(1-p)\,p\,\rho + p^2\,(-3 + 15\,p - 16\,p^2 + 5\,p^3)\,\rho^2\right) \\
&\quad - (1-p)\,p\,z^2\,\rho\left(-2 - (-2+p)\,p\,(-2+5\,p)\,\rho + p^2\,(-6 + 21\,p - 26\,p^2 + 10\,p^3)\,\rho^2\right. \\
&\qquad\qquad \left. - (1-p)\,p^3\,(2 - 15\,p + 23\,p^2 - 13\,p^3 + 2\,p^4)\,\rho^3\right) \\
&\quad + (1-p)\,p^2\,z^3\,\rho^2\left(-1 + p + p\,(-6 + 23\,p - 26\,p^2 + 10\,p^3)\,\rho + p^2\,(-5 + 28\,p - 60\,p^2 + 52\,p^3 - 20\,p^4 + 2\,p^5)\,\rho^2\right. \\
&\qquad\qquad - p^3\,(6 - 35\,p + 86\,p^2 - 123\,p^3 + 91\,p^4 - 32\,p^5 + 4\,p^6)\,\rho^3 \\
&\qquad\qquad \left. + p^4\,(2 - 11\,p + 14\,p^2 + 4\,p^3 - 32\,p^4 + 36\,p^5 - 17\,p^6 + 3\,p^7)\,\rho^4\right) \\
&\quad - (1-p)^3\,p^4\,z^4\,\rho^4\left(-3 + 7\,p - 5\,p^2 - p\,(6 - 29\,p + 41\,p^2 - 24\,p^3 + 4\,p^4)\,\rho\right. \\
&\qquad\qquad - (1-p)\,p^2\,(1 - 8\,p + 26\,p^2 - 21\,p^3 + 5\,p^4)\,\rho^2 \\
&\qquad\qquad - p^3\,(2 - 3\,p + 9\,p^2 - 27\,p^3 + 41\,p^4 - 26\,p^5 + 6\,p^6)\,\rho^3 \\
&\qquad\qquad \left. + p^4\,(3 - 15\,p + 35\,p^2 - 41\,p^3 + 28\,p^4 - 11\,p^5 + 2\,p^6)\,\rho^4\right) \\
&\quad + (1-p)^4\,p^6\,z^5\,\rho^6\left((1-p)\,(-1+2\,p)\,(3 - 5\,p + p^2) + p\,(-2 + 7\,p - 15\,p^2 + 25\,p^3 - 20\,p^4 + 6\,p^5)\,\rho\right. \\
&\qquad\qquad - p^2\,(-2 + 3\,p + 10\,p^2 - 22\,p^3 + 23\,p^4 - 12\,p^5 + 3\,p^6)\,\rho^2 \\
&\qquad\qquad + p^4\,(-7 + 33\,p - 43\,p^2 + 27\,p^3 - 8\,p^4 + p^5)\,\rho^3 \\
&\qquad\qquad \left. + p^4\,(1 - 2\,p - 7\,p^2 + 14\,p^3 - 9\,p^4 + 2\,p^5)\,\rho^4\right) \\
&\quad + (1-p)^6\,p^8\,z^6\,\rho^8\left(1 - 6\,p + 11\,p^2 - 8\,p^3 + 3\,p^4 - p^2\,(-5 + 12\,p - 6\,p^2 + p^3)\,\rho\right. \\
&\qquad\qquad \left. - (-1+p)\,p^2\,(-1 - p + 5\,p^2)\,\rho^2 + p^5\,(1+p)\,\rho^3 - p^5\,\rho^4\right)
\end{aligned}
$$

This would allow us to compute the reliability for, say, $p = 0.871$, $\rho = 0.913$, and large n with an excellent accuracy (for very large values of n, it would be best to perform in partial fraction decomposition first, in order to isolate the prevailing eigenvalue).

2.2 Location of the complex zeros

Following (Tanguy 2007a), we have been able to plot the complex zeros of the two-terminal reliability polynomial as a function of p, for each value of ρ. Such a study is a familiar one for people involved in graph polynomials, because it could lead to insight to properties or symmetries of the solution (Biggs 1993; Brown and Colbourn 1992; Chang and Shrock 2003; Gordon and McMahon 2001).

When ρ decreases from 1 to 0, the structure of the zeros is distorted and expands (a diaporama will be presented). Special features occur at critical values of ρ. While the observed structures are clearly reminiscent of those obtained for the simpler “double fan” configuration. A huge difference exists, however. Whereas in the former situation, the limiting curves where connected, we have here a transition between two regimes. For $\rho > \rho_c \approx 0.068872$, the limiting structure of the zeros is indeed connected. Yet, below ρ_c, the limiting structure is made of two distinct parts. We have displayed in Figs. 2 and 3 what happens at ρ_c.

3 Towards further exact solutions for more complex configurations

A clever algorithm was proposed in (Carlier and Lucet(1996)) to compute the two-terminal reliability of the Manhattan $3 \times n$ grid in linear time (with respect to n). A few years ago, this was vindicated by another paper mentioning the importance of the tree-width of the graph (Wolle 2003). Quite recently, Hardy and collaborators (2007) have indicated that they are able to routinely compute the two-terminal reliability for Manhattan grids such as 8, 8, etc. (see Table III) for which all edges are identical. These graphs are rather wide, and it would be very interesting to know the exact generating function of the true solution.

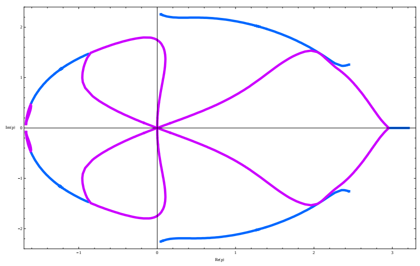

Figure 2: Limiting curves for the graph of Fig. 1 for $\rho = 0.068872$ (in purple when 1 is one of the two roots of maximum modulus, and in blue otherwise).

Our suggestion is to adapt their calculations or similar ones, knowing that the true result must follow a recursion of unknown but *finite* order. The main steps of the procedure are

1. Adapt the software to multi-precision calculations (packages exist). Even though, for all practical purpose, it is not necessary to go beyond two or three digits of accuracy, we need to obtain the exact result when p is given.

2. Use rational values for p when calculating the two-terminal reliability for $n = 2, 3, 4, ..., n_{\max}$: the results must be rational, too. Identify each rational (several hundred digits of accuracy may be helpful in some cases)

3. Knowing $\mathrm{Rel}_2(n, p)$ for $n = 2, 3, 4, ...n_{\max}$ and a given p, compute the (m, m) Padé approximant of the series. If $n_{\max}$ is large enough, the Padé approximant should remain stable at some point (one might consider $p = 1/2$ for a first test). The degree of the numerator of the rational fraction may be equal, or less than that of the denominator: unless an accidental degeneracy occurs, the degrees of numerator and denominator do not change with the choice of p. The variation of these degrees with respect to the width of the grid could be a very interesting information.

4. The identification of the true denominator (respectively, numerator) can be done using different values of p, for instance $\{\frac{1}{100}, \frac{2}{100}, \cdots, \frac{99}{100}\}$ through polynomial regressions. Coefficient of lowest and highest degrees should have very simple expressions in p, while all of them would mainly exhibit very small *integer* coefficient.

5. We have the exact solution for the generating function $\mathcal{G}$. Any further calculation would best be performed using $\mathcal{G}$.

As a short example, we could consider the reliability of the recursive family of graph studied above.

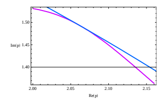

Figure 3: Tangency of the two sets of limiting curves at ρ_c.

Using $p = \frac{1}{2}$ (and $\rho = 1$), we would quickly find

$$\mathcal{G}(z, p = \frac{1}{2}) = \frac{z\,\left(-131072 - 53248\,z + 22528\,z^2 - 1088\,z^3 - 16\,z^4 + z^5\right)}{(-1+z)\,\left(1048576 - 1277952\,z + 454656\,z^2 - 66496\,z^3 + 4224\,z^4 - 114\,z^5 + z^6\right)} \tag{1}$$

obtaining in this way the degrees of numerator and denominator, and allowing for a possible decrease of $n_{\max}$. Repeating the calculation for $p \in \{\frac{1}{100}, \frac{2}{100}, \cdots, \frac{99}{100}\}$ would ensure that each coefficient is properly identified as a polynomial in p, as it should. The coefficient of z^2 of the denominator would read $(1-p)^2\,p\,\left(2 + 6\,p - 14\,p^3 - 5\,p^4 + 23\,p^5 - 13\,p^6 + 2\,p^7\right)$ while that of z^6 would be $(1-p)^{12}\,p^8$ (we assume that the coefficient of z^0 of the denominator is equal to 1).

4 Conclusion

We have shown that it is possible to find the exact two-terminal reliability for arbitrarily large networks. The solution can be expressed as a product of transfer matrices. We have also observed that the "parallel" character of the graph does not imply that the location of the complex zeros of the reliability polynomial is a 1-connected structure. Finally, we propose a method to determine the generating function of the exact solutions for more complicated recursive structures; this procedure requires only modest changes to existing softwares and various polynomial interpolations, making full use of the knowledge of what the true solution must be.

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