Survival analysis in nonhomogeneous branching random walks

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Abstract

The properties of BRW depending on intensity of a branching process at the source are investigated: a supercritical branching process at the source in virtue of transiency of a random walk on the higher dimensions of lattice $\mathbf{Z}^d$ $(d \geq 3)$ may activate both supercritical, critical and even subcritical BRW. Critical and subcritical branching processes at the source on the higher dimensions of lattices can entail only subcritical BRW. Classification BRW based on the asymptotic behavior of the conditional expectation of population size and the number of particles at an arbitrary lattice point is obtained. Applications of BRW in reliability theory are discussed.

1 Introduction

As known (Gnedenko *et al.*, 1965) birth and death processes have found applications in reliability theory. Models describable in terms of the birth, death and walk of particles on multidimensional lattices can be useful too to study complex reserve systems with regeneration of their elements. Possibility of application of such models in reliability theory explains the interest in continuous-time branching random walks (BRW) on lattices with one source of branching. For example, in (Vatutin *et al.*, 2003) a modification of critical BRW on $\mathbf{Z}$ was used to analyze properties of a queueing system with countable number of servers. One of the main problems in BRW is the investigation of asymptotic behavior of the particle population and the number of particles at an arbitrary point on the lattice. For example, in models of reserve systems with regeneration of their elements the number of particles at the source on the lattice may be treated as the number of serviceable elements in the system, the number of particles outside the source — as renewable elements, the dead particles at the source — as nonrecoverable elements and the particle population — as the number of serviceable and renewable elements together. The aim of the paper is to formulate results for BRW, which might be useful in reliability theory. First of all, one of the main problems in such models is the investigation of survival probabilities of the particle population and the number of particles at the source. The asymptotic behavior of survival probabilities for BRW was studied in (Yarovaya, 2005). In the present paper we investigate the properties of BRW depending on intensity of a branching process at the source: a supercritical branching process at the source in virtue of transiency of a random walk on the higher dimensions of lattice $\mathbf{Z}^d$ $(d \geq 3)$ may activate both supercritical, critical and even subcritical BRW. Critical and subcritical branching processes at the source on the higher dimensions of lattices can entail only subcritical BRW. The particular attention in this paper will be paid to critical and subcritical BRW and their dependence on the lattice dimension. The evolution of the processes are essentially dependent on the structure of the environment where the walk takes place. As it has been shown in (Bogachev and Yarovaya, 1998a) and (Yarovaya, 2007) it is possible to study in detail influence of nonhomogeneity and noncompactness, as well as that of dimensionality of a space of walk, on the properties of the BRW. Classification based on the asymptotic behavior of the conditional expectation of population size and the number of particles at an arbitrary lattice point is obtained.

2 Description of BRW

We study evolution of system of particles on $\mathbf{Z^d}$ $(d \geq 1)$ whose state is described by the number $\mu_t(y)$ of particles at each point $y \in \mathbf{Z^d}$ and particle population $\mu_t = \sum_{y\in\mathbf{Z^d}}$ size at the time t. The population of individuals is initiated at time $t = 0$ by a single particle at the point x, that is, $\mu_0(y) = \delta_x(y)$. The random walk of particles obeys completely the infinitesimal transition matrix $A = \|a(x, y)\|_{x,y\in\mathbf{Z}^d}$ and is assumed to be symmetric, $a(x, y) = a(y, x)$, homogeneous, $a(x, y) = a(0, y - x) = a(y - x)$, irreducible, regular, $\sum\limits_{x\in\mathbf{Z}^d} a(x) = 0$ with $a(x) \geq 0, \quad x \neq 0, \quad a(0) < 0$, and having a finite variance of jumps, $\sum\limits_{x\in\mathbf{Z}^d} x^2 a(x) < \infty$. In virtue of symmetry and homogeneity of the random walk, the conditions $\sum_{y\in\mathbf{Z}} a(x, y) = 0$ and $\sum_{x\in\mathbf{Z}} a(x, y) = 0$ are satisfied for the matrix A. At the source of branching (at the origin in our model) the branching process is defined by the infinitesimal generating function $f(u) := \sum_{n=0}^{\infty} b_n u^n$, where $b_n \geq 0$ for $n \neq 1$, $b_1 < 0$ and $\sum_n b_n = 0$. We assume that $\beta_r = f^{(r)}(1) < \infty, \quad r \in \mathbf{N}, \beta = \beta_1 = f^{'}(1)$. Therefore, if $\mu_t(0) > 0$ particles were at the time instant t at the origin, then, independently of the rest of particles each particle in the time $[t, t + h)$ can either jump with the probability $p(h, 0, y) = a(y)h + o(h)$ to the point $y \neq 0$, or produce particles $(n \neq 1)$including itself, or die (the case of $n = 0$) with the probability $p_*(h, n) = b_n h + o(h)$, or survive (no changes) with the probability $1 - \sum_{y\neq0} a(y)h - \sum_{n\neq1} b_n h + o(h)$. The standard method can be used to prove that the time of particle stay at the source is distributed exponentially with the parameter $a(0) + b_1$.

Under these conditions, the random walk transition probability has the asymptotics $p(t, x, y) \sim \gamma_d \cdot t^{-d/2}$ as $t \to \infty$, where γ_d is a constant depending on the space dimension (Yarovaya, 2007). Denote by $G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t, x, y)\, dt$ the Green function of the random walk, that is the Laplace transform of $p(t, x, y)$ with respect to t. From this $G_\lambda(0, 0)|_{\lambda=0}$ is finite for $d \geq 3$. Therefore, the random walk is transient if $d \geq 3$ and recurrent otherwise. Put $\beta_{\mathrm{C}} := 1/G_0(0, 0)$, then $\beta_{\mathrm{c}} = 0$ for $d = 1, 2$ and $\beta_{\mathrm{C}} > 0$ for $d \geq 3$.

Suppose that $\beta_r := f^{(r)}(u)|_{u=1} < \infty$ for all $r \in \mathbf{N}$. The point β_{C} is critical since the asymptotic behavior of the process is essentially different for $\beta_1 > \beta_{\mathrm{C}}$, $\beta_1 = \beta_{\mathrm{C}}$, and $\beta_1 < \beta_{\mathrm{C}}$. Let $m_n(t, x, y) := E_x \mu_t^n(y)$ and $m_n(t, x) := E_x \mu_t^n$, $(n \in \mathbf{N})$, where E_x denotes the mathematical expectation under the condition $\mu_0(\cdot) = \delta_x(\cdot)$. The longtime asymptotics of all the moments (of integer orders) for the μ_t as well as for the $\mu_t(y)$ have been studied in (Bogachev and Yarovaya, 1998a). If $\beta_1 > \beta_{\mathrm{C}}$, then in the sense of convergence of moments under the normalization $e^{-\lambda_0 t}$ (where exponent λ_0 is determined from the equation $\beta_1 G_\lambda(0, 0) = 1$), the random variables $\mu_t(y)$ and μ_t have a limit distribution as $t \to \infty$ (Bogachev and Yarovaya, 1998b).

In the case $\beta_1 = \beta_{\mathrm{C}}$, the growth of the moments for the μ_t and $\mu_t(y)$ appears to be irregular with respect to the number n. This means that the behavior of the random variable μ_t and $\mu_t(y)$ as $t \to \infty$ substantially differs from the behavior of the moments.

For this reason there is found the asymptotic behavior of the survival probability of the process $Q(t, x)$ at time t and the probability $Q(t, x, 0)$ that the number of individuals at the origin at time t is positive. Put $Q(t, 0) = Q(t)$ and $Q(t, 0, 0) = q(t)$. The following results for arbitrary dimension d are given for the critical branching random walk started at the origin (Yarovaya, 2005):

THEOREM 1. *If $\beta = \beta_c$, then $q(t) \sim k_d u(t)$ and $Q(t) \sim K_d v(t)$, as $t \to \infty$, where k_d, K_d are positive constants, and functions u, v have the forms:*

$$
\begin{array}{lll}
u(t) = t^{-1/2}(\ln t)^{-1}, & v(t) = t^{-1/4} & d = 1; \\
u(t) = t^{-1}, & v(t) = (\ln t)^{-1/2} & d = 2; \\
u(t) = t^{-1/2}(\ln t)^{-1}, & v(t) \equiv 1 & d = 3; \\
u(t) = t^{-1}(\ln t), & v(t) \equiv 1 & d = 4; \\
u(t) = t^{-1}, & v(t) \equiv 1 & d \geq 5.
\end{array}
$$

The case of the dimension $d = 1, 2$ was studied in detail by (Yarovaya, 2009).

3 Main Results

Instead of cumbersome wording of theorems basic results are represented in two tables. As $t \to \infty$, the moments $E\mu_t(y)$ and $E\mu_t$ have the asymptotics (Yarovaya, 2007)

$$E_x\mu_t(y) \sim C_d(x,y)\, u(t), \qquad E_x\mu_t \sim C_d(x)\, v(t),$$

where the functions u, v are of the form shown in the next Table.

Table 1: Classification of BRW depending on intensity of a source.

Branching process at the source	**Random Walk**	**Branching Random Walk**	$u(t)$	$v(t)$
Supercritical $\beta > 0$	recurrent $d = 1, 2,\ \beta_c = 0$	Supercritical $\beta > \beta_c$	$e^{\lambda}t$	$e^{\lambda}t$
Supercritical $\beta > 0$	transient $d \geq 3,\ \beta_c > 0$	Supercritical $\beta > \beta_c$	$e^{\lambda}t$	$e^{\lambda}t$
Supercritical $\beta > 0$	transient $d = 3,\ \beta_c > 0$	Critical $\beta = \beta_c$	$1/\sqrt{t}$	$\sqrt{t}$
Supercritical $\beta > 0$	transient $d = 4,\ \beta_c > 0$	Critical $\beta = \beta_c$	$1/\ln t$	$t/\ln t$
Supercritical $\beta > 0$	transient $d \geq 5,\ \beta_c > 0$	Critical $\beta = \beta_c$	1	t
Supercritical $\beta > 0$	transient $d \geq 3,\ \beta_c > 0$	Subcritical $\beta_c > \beta > 0$	$t^{-d/2}$	1
Critical $\beta = 0$	recurrent $d = 1,\ \beta_c = 0$	Critical $\beta = \beta_c$	$1/\sqrt{t}$	1
Critical $\beta = 0$	recurrent $d = 2,\ \beta_c = 0$	Critical $\beta = \beta_c$	$1/t$	1
Critical $\beta = 0$	transient $d \geq 3,\ \beta_c > 0$	Subcritical $\beta_c > \beta = 0$	$t^{-d/2}$	1
Subcritical $\beta < 0$	recurrent $d = 1,\ \beta_c = 0$	Subcritical $\beta < \beta_c$	$t^{-3/2}$	$1/\sqrt{t}$
Subcritical $\beta < 0$	recurrent $d = 2,\ \beta_c = 0$	Subcritical $\beta < \beta_c$	$(t \ln^2 t)^{-1}$	$1/\ln t$
Subcritical $\beta < 0$	transient $d \geq 3,\ \beta_c > 0$	Subcritical $\beta < \beta_c$	$t^{-d/2}$	1

The asymptotic behavior of the conditional expectations of the number of particles at an arbitrary lattice point and population size is obtained using Theorem I and results presented in the Table I. As $t \to \infty$, the moments $E_x[\mu_t(y)|\mu_t(y) > 0]$ and $E_x[\mu_t|\mu_t > 0]$ have the representations

$$E_x[\mu_t(y)|\mu_t(y) > 0] \sim K_d(x,y)\, u^*(t), \qquad E_x[\mu_t|\mu_t > 0] \sim K_d(x)\, v^*(t).$$

where the functions u^*, v^* are of the form:

Table 2: Asymptotic behavior of conditional expectations as $t \to \infty$ in BRW.

Branching Random Walk		**Dimension of the Lattice**	$u^*(t)$	$v^*(t)$
Supercritical	$\beta > \beta_c$	$d \geq 1$	$e^{\lambda t}$	$e^{\lambda t}$
Critical	$\beta = \beta_c$	$d = 1$	$\ln t$	$t^{1/4}$
		$d = 2$	1	$\sqrt{\ln t}$
		$d = 3$	$\ln t$	$\sqrt{t}$
		$d = 4$	$t/\ln^2 t$	$t/\ln t$
		$d \geq 5$	t	t
Subcritical	$\beta < \beta_c$	$d = 1$	$1/\sqrt{t}$	1
		$d = 2$	$1/\ln t$	1
		$d \geq 3$	1	1

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References

Albeverio, S., Bogachev, L., Yarovaya, E. (1998). Asymptotics of branching symmetric random walk on the lattice with a single source. *Comptes Rendus de l'Academie des Sciences, Paris, series I. 326*, 975–980.

Bogachev, L., Yarovaya, E. (1998a). Moment analysis of a branching random walk on a lattice with a single source. *Doklady Akademii Nauk 363*, 439–442.

Bogachev, L., Yarovaya, E. (1998b). A limit theorem for supercritical branching random walk on $\mathbf{Z^d}$ with a single ource. *Russian Mathematical Surveys 53*, 1086–1088.

Gnedenko, B., Belyaev, Yu., Solov'ev, A. (1965) *Mathematical Methods in Reliability Theory.* Nauka, Moscow (in Russian).

Vatutin, V., Topchii, V., Yarovaya, E. (2003). Catalytic branching random walk and queueing systems with random number of independent servers *Teoriya Jmovirnostei ta Matematichna Statistika 69*, 158–172.

Yarovaya, E. (2009). A limit theorem for critical branching random walk on $\mathbf{Z^d}$ with a single source. *Russian Mathematical Surveys*, *60*, 1, 175–176.

Yarovaya, E. (2009). Critical branching random walks on low dimensional lattices. *Discrete Mathematics 21*, 117–138.

Yarovaya, E. (2007). *Branching Random Walks in Inhomogeneous Medium.* MSU, Moscow (in Russian).