Noname manuscript No.
(will be inserted by the editor)

Deterioration Processes With Increasing Thresholds

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Received: date / Accepted: date

Abstract The present paper derives the reliability functions and hazard functions, when the threshold for deterioration is an increasing function of time. Four cases are considered. Case I: The threshold is a step function with K jumps at known points. Case II: The threshold is a step function with K jumps, where the location of jumps are random, following a renewal process. Case III: The controlled case. The jumps occur after the first crossing of a control limit. Case IV: The threshold increases linearly. The theory is illustrated in all four cases with a Markovian deterioration process, i.e., a compound Poisson process with i.i.d. jumps following an exponential distribution.

Keywords Compound Poisson Process; Deterioration Process; Hazard Function; Increasing Threshold Function; Reliability Function

Mathematics Subject Classification (2000) 60J55 · 60J75 · 62N05

1 Introduction

The present paper is a fourth one, in a series of papers on the reliability (availability) of systems subjected to compound Poisson deterioration processes. In the first paper (Zacks, 2004) non-homogeneous compound Poisson processes were considered. Failure time distributions and their moments were studied. The second paper (Zacks, 2006) discussed failure distributions associated with general compound renewal processes. The third paper (Zacks, 2009) studies the availability of systems which are replaced after failure, either immediately or after a random time. In the present paper, the failure times are the first crossing of a deterioration process $\{D(t), t \geq 0\}$ of an increasing boundary, $B(t)$, where $B(0) = \beta_0$, $0 < \beta_0 < \infty$. More specifically, let $D(t) = \sum_{n=0}^{N(t)} X_n$ denote a deterioration process, where $\{N(t)\}$ is a Poisson process, and $X_1, X_2, \ldots$ are

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i.i.d. random variables, representing the amount of damage to the system ($X_n > 0, n \geq 1$) in each shock event. The system fails at the stopping time

$$T = \inf\{t > 0 : D(t) \geq B(t)\}. \tag{1}$$

There are various applications of the present theory. We can think of a bridge that, once in a while is strengthened. Another application is in insurance. For a given insurance file, let $B(t)$ be the total amount of funds entering the system (premiums, etc.) up to time t, and $D(t)$ are the total claims (minus deductibles) up to time t. The stopping time T is called in insurance a ruin time. The reliability of a system at time t is $R(t) = P\{T > t\}$. The corresponding hazard function is $\Lambda(t) = -\frac{d}{dt}R(t)/R(t)$. We consider several cases.

Case I:

The first case is of an increasing step-function

$$B(t) = \sum_{n=1}^{K} I\{\tau_{i-1} < t \leq \tau_i\}\beta_{i-1} + I(\tau_K < t)\beta_K, \tag{2}$$

where $0 < \beta_0 < \beta_1 < \cdots < \beta_K$, τ_i, $i = 1, 2, \ldots, K$, are fixed growth times, with $\tau_0 = 0$.

Case II:

The second case holds when $\{\tau_i, i \geq 1\}$ are random times following a renewal process.

Case III:

The third case is that of controlled threshold growth. In this case

$$\tau_i = \inf\{t > 0 : D(t) \geq \gamma_i\}, \quad i = 1, 2, \ldots \tag{3}$$

where $\gamma_i < \beta_i$, and $\gamma_i < \gamma_{i+1}$ $(i = 1, 2, \ldots)$.

Notice that the control limits are below the threshold levels. The instant $D(t)$ crosses a control limit, γ, the threshold level, β, is increased.

Case IV:

The fourth case is that of linearly increasing threshold $B(t) = \beta_0 + \gamma t$, where $0 < \beta_0 < \infty$ and $0 < \gamma < \infty$. This is the usual case in insurance ruin problems.

We start with the general theory for each case, followed by the special case of exponential damage.

2 Preliminaries

In the present paper we focus attention on processes $\{D(t), t \geq 0\}$ which are compound Poisson, i.e., $D(t) = \sum_{n=0}^{N(t)} X_n$, $t \geq 0$, where $\{N(t), t \geq 0\}$ is a homogeneous Poisson process; $X_0 \equiv 0$; $X_1, X_2, \ldots$ are independent identically distributed (i.i.d.) random variables, independent of $\{N(t); t \geq 0\}$. We further assume that X_n have a common absolutely continuous distribution, F, $F(0) = 0$, with density function f. Let $H(y; t) = P\{D(t) \leq y\}$ be the distribution function of $D(t)$. H has an atom at $y = 0$, i.e., $H(0; t) = e^{-\lambda t}$, where λ is the intensity coefficient of $\{N(t), t \geq 0\}$. Moreover, for $0 < y < \infty$,

$$H(y; t) = \sum_{n=0}^{\infty} p(n; \lambda t) F^{(n)}(y), \tag{4}$$

where $p(n;\lambda t) = e^{-\lambda t}(\lambda t)^n/n!$, $n \geq 0$, is the probability mass function of the Poisson distribution Pois(λt).

$F^{(n)}(y) = \int_0^y f(x)F^{(n-1)}(y-x)dx$, all $y > 0$, $n \geq 1$, where $F^{(0)}(y) = 1$ for all $y > 0$. This is the n-fold convolution of F. The density function of $D(t)$, for $y \in (0, \infty)$ is

$$h(y;t) = \sum_{n=1}^{\infty} p(n;\lambda t) f^{(n)}(y), \tag{5}$$

where $f^{(n)}(y)$, $n \geq 1$, is the n-fold convolution of f. Notice that $f^{(n)}(y) = 0$ for all $y \leq 0$ and for $y > 0$, $f^{(n)}(y) = \int_0^y f(x)f^{(n-1)}(y-x)dx$, $n \geq 2$, with $f^{(1)}(y) = f(y)$, $y > 0$.

The process $\{D(t), t \geq 0\}$ is a strongly Markovian jump process, where the epochs of jump are the arrival times of the Poisson renewal points. Let T be a stopping time defined in Eq. (1). Define the defective density

$$g(y;t) = \frac{d}{dy} P\{D(t) \leq y, T > t\}, \quad 0 < y < B(t). \tag{6}$$

Since $B(t) \geq \beta_0$ for all t,

$$g(y;t) = h(y;t) \text{ for all } 0 < y \leq \beta_0. \tag{7}$$

The system reliability, when the threshold is $\{B(t), t \geq 0\}$ is

$$R(t) = P\{T > t\} = e^{-\lambda t} + \int_0^{B(t)} g(y;t)dy. \tag{8}$$

The corresponding hazard function is

$$\begin{aligned} \Lambda(t) &= -\frac{d}{dt}R(t)/R(t) \\ &= \left[\lambda e^{-\lambda t} + B'(t)g(B(t);t) + \int_0^{B(t)} \left(\frac{\partial}{\partial t} g(y;t)\right) dy\right] \Big/ R(t), \end{aligned} \tag{9}$$

where $B'(t) = \frac{d}{dt}B(t)$. Notice that if $B(t)$ is the step function (2) then $B'(t) = 0$ all $t \neq \tau_i$. A specific hazard function for this case will be developed in the next section.

3 The Reliability and Hazard in Case I

In Case I the threshold is a step-function. In the case of $K = 0$ the threshold is a constant β_0, and then the reliability function is

$$R(t;\beta_0) = H(\beta_0;t). \tag{10}$$

This is obviously a decreasing function of t on $(0, \infty)$. The hazard function is then

$$\begin{aligned} \Lambda(t;\beta_0) &= -\frac{d}{dt}H(\beta_0;t)/H(\beta_0;t) \\ &= \lambda \sum_{n=0}^{\infty} p(n;\lambda t)(F^{(n)}(\beta_0) - F^{(n+1)}(\beta_0))/H(\beta_0;t). \end{aligned} \tag{11}$$

Generally, when $K \geq 1$

$$B(t;\boldsymbol{\tau},\boldsymbol{\beta}) = \sum_{i=1}^{K} I(\tau_{i-1} < t \leq \tau_i)\beta_{i-1} + I(\tau_K < t)\beta_K. \tag{12}$$

Define recursively the defective density $g(t;\boldsymbol{\tau},\boldsymbol{\beta})$ as follows:

$$g(x;t) = h(x;t)I(x \leq \beta_0), \quad \text{if } 0 < t \leq \tau_1. \tag{13}$$

For $i \geq 2$,

$$\begin{aligned} g(x;t) = \sum_{i=2}^{\infty} I(\tau_{i-1} < t \leq \tau_i) \int_0^{x \wedge (\beta_{i-2})} g(y;\tau_{i-1}) \, \cdot \\ \cdot \;\; h(x-y;t-\tau_{i-1})dy \cdot I(0 < x \leq \beta_{i-1}). \end{aligned} \tag{14}$$

Notice that if $x \leq \beta_0$ then $g(x;t) = h(x;t)$ for all $t > 0$.

The reliability function is given by

$$\begin{aligned} R_K(t;\boldsymbol{\tau},\boldsymbol{\beta}) &= I(t \leq \tau_1)H(\beta_0;t) \\ &+ \sum_{i=1}^{K-1} I(\tau_i < t \leq \tau_{i+1}) \Big[e^{-\lambda\tau_i} H(\beta_i; t-\tau_i) \\ &+ \int_0^{\beta_{i-1}} g(x;\tau_i)H(\beta_i - x; t-\tau_i)dx \Big] \\ &+ I(t > \tau_K) \Big[e^{-\lambda\tau_K} H(\beta_K; t-\tau_K) \\ &+ \int_0^{\beta_{K-1}} g(x;\tau_K)H(\beta_K - x; t-\tau_K)dx \Big]. \end{aligned} \tag{15}$$

The hazard function is

$$\Lambda_K(t;\boldsymbol{\tau},\boldsymbol{\beta}) = \frac{-\frac{d}{dt}R_K(t;\boldsymbol{\tau},\boldsymbol{\beta})}{R_K(t;\boldsymbol{\tau},\boldsymbol{\beta})}.$$

Let

$$\begin{aligned} D_0(t;\gamma) &= -\frac{d}{dt}H(\gamma;t) \\ &= \lambda \sum_{n=0}^{\infty} p(n;\lambda t)[F^{(n)}(\gamma) - F^{(n+1)}(\gamma)]. \end{aligned} \tag{16}$$

Thus we obtain $-\dfrac{d}{dt}R_K(t;\boldsymbol{\tau},\boldsymbol{\beta})$ by substituting in (15) the function $D_0(t,\beta_0)$ corresponding to $H(\beta_0,t)$; the function $D_0(t-\tau_i;\beta_i)$ corresponding to $H(\beta_i;t-\tau_i)$; etc.

4 The Reliability and Hazard in Case II

In Case II, the K jump points of $B(t)$ are randomly assigned according to some renewal process. More specifically, let $U_1, U_2, \ldots, U_K$ be i.i.d. positive random variables, representing the times elapsing between the jumps of $B(t)$; i.e., $\tau_1 = U_1$, $\tau_2 = U_1 + U_2, \ldots$, $\tau_K = U_1 + \cdots + U_K$. Let $\psi(u)$ denote the density of U, on $(0,\infty)$. The reliability function is $R^*_K(t;\beta_0,\beta_1,\ldots,\beta_K)$. This function is defined recursively as follows.

(i) For $K = 1$, let

$$R_1^*(t;\beta_0,\beta_1) = P(U > t)H(\beta_0;t) + \int_0^t \psi(u)\Big[e^{-\lambda u}H(\beta_1;t-u) + \int_0^{\beta_0} h(y;u)H(\beta_1 - y;t-u)dy\Big]du. \tag{17}$$

For each $n = 2,\ldots,K$, define

$$R_n^*(t;\beta_0,\beta_1,\ldots,\beta_n) = P(u > t)H(\beta_0;t) + \int_0^t \psi(u)\Big[e^{-\lambda u}R_{n-1}^*(t-u;\beta_1,\ldots,\beta_n) + \int_0^{\beta_0} h(y;u)R_{n-1}^*(t-u;\beta_1 - y,\ldots,\beta_n - y)dy\Big]du. \tag{18}$$

General solution of this integral difference equation is complicated, we will develop later some special cases.

The hazard function corresponding to $R_1^*(t;\beta_0,\beta_1)$ is

$$\Lambda_1(t;\beta_0,\beta_1) = \Big\{P(U > t)D_0(t;\beta_0) + \int_0^t \psi(u)\Big[e^{-\lambda u}D_0(t-u;\beta_1) + \int_0^{\beta_0} h(y;u)D_0(t-u;\beta_1 - y)dy\Big]du\Big\} \div R_1^*(t;\beta_0,\beta_1). \tag{19}$$

For $n = 1,\ldots,K$, define

$$D_n(t;\beta_0,\ldots,\beta_n) = P(U > t)D_0(t;\beta_0) + \int_0^t \psi(u)\Big[e^{-\lambda u}D_{n-1}(t-u;\beta_1,\ldots,\beta_n) + \int_0^{\beta_0} h(y;u)D_{n-1}(t-u;\beta_1 - y,\ldots,\beta_n - y)dy\Big]du. \tag{20}$$

Then, the hazard function of the system is

$$\lambda_K(t;\beta_0,\ldots,\beta_K) = \frac{D_K(t;\beta_0,\ldots,\beta_K)}{R_K^*(t;\beta_0,\ldots,\beta_K)}. \tag{21}$$

5 Reliability and Hazard in Case III

In Case III the jump from β_0 to a larger level is at the first time the process $\{D(t), t \geq 0\}$ crosses a horizontal control line at level γ_0, $\gamma_0 < \beta_0$. The jump point is at

$$T_c = \inf\{t > 0 : D(T) \geq \gamma_0\}. \tag{22}$$

Let $R(T_c) = D(T_c) - \gamma_0$. If $D(T_c) \geq \beta_0$ the system fails at time T_c. If $D(T_c) < \beta_0$ the system survives at T_c, and the threshold increases immediately to β_1. The joint density of $(T_c, R(T_c))$ is

$$p(t, r; \gamma_0) = \lambda e^{-\lambda t} f(\gamma_0 + r) + \lambda \int_0^{\gamma_0} h(y; t) f(\gamma_0 + r - y) dy. \tag{23}$$

The marginal density of T_c is

$$\begin{aligned} \psi_c(t; \gamma_0) &= \int_0^{\infty} p(t, r; \gamma_0) dr \\ &= \lambda e^{-\lambda t} \bar{F}(\gamma_0) + \lambda \int_0^{\gamma_0} h(y; t) \bar{F}(\gamma_0 - y) dy, \end{aligned} \tag{24}$$

where $\bar{F}(x) = 1 - F(x)$. We formulate now the reliability function for the simpler case of one increase of the threshold, i.e.,

$$B_c(t) = I(t \leq T_c)\beta_0 + I(t > T_c)\beta_1, \tag{25}$$

where $0 < \beta_0 < \beta_1$. The system fails as soon as $D(t) \geq B_c(t)$. Thus, the conditional reliability given T_c is

$$R_c(t \mid T_c) = I(t \leq T_c) + I(t \geq T_c, D(T_c) < \beta_0) H(\beta_1 - D(T_c); t - T_c). \tag{26}$$

Hence, the reliability function is

$$R_c(t; \gamma_0, \beta_0, \beta_1) = H(\gamma_0; t) + \int_0^t \int_0^{\beta_0 - \gamma_0} p(s, r; \gamma_0) H(\beta_1 - \gamma_0 - r; t - s) dr ds. \tag{27}$$

The corresponding hazard function is

$$\begin{aligned} \Lambda_c(t; \gamma_0, \beta_0, \beta_1) = &\left\{ D_0(t; \gamma_0) + \int_0^t \int_0^{\beta_0 - \gamma_0} p(s, r; \gamma_0) D_0(t - s; \beta_1 - \gamma_0 - r) dr ds \right\} \\ &\div R_c(t; \gamma_0, \beta_0, \beta_1). \end{aligned} \tag{28}$$

6 Reliability and Hazard in Case IV

In Case IV the threshold boundary is

$$B(t) = \beta + \gamma t, \quad 0 \leq t < \infty; \quad 0 < \gamma < \infty, \quad 0 < \beta < \infty. \tag{29}$$

Without loss of generality we can assume that $\gamma = 1$. Indeed, if $\gamma \neq 1$ make the transformation $\lambda' = \lambda \cdot \gamma$ and $t' = \frac{t}{\gamma}$. Then $p(n; \lambda' t) = p(n; \lambda t)$ for all $n \geq 0$. Consider the stopping time

$$T(\beta) = \inf\{t > 0 : D(t) \geq \beta + t\}. \tag{30}$$

This is the failure time of the system. As in (8),

$$R_L(t; \beta) = e^{-\lambda t} + \int_0^{\beta + t} g(x; t, \beta) dx. \tag{31}$$

As shown in Zacks (2005), the defective density $g(x;t,\beta)$ is given in terms of the density $h(x;t)$ as

$$\begin{aligned} g(x;t,\beta) &= h(x;t)I(x \le \beta) \\ &\quad + I(\beta < x \le \beta + t)\Big[h(x;t) - e^{-\lambda(\beta+t-x)}h(x;x-\beta) \\ &\quad - (\beta+t-x)\int_0^{x-\beta} h(u+\beta;u)\frac{1}{t-u}h(x-\beta-u;t-u)du\Big]. \end{aligned} \tag{32}$$

Accordingly, the reliability function is

$$\begin{aligned} R_L(t;\beta) &= H(\beta+t;t) - \int_\beta^{\beta+t} e^{-\lambda(\beta+t-x)} \cdot \\ &\quad \cdot\ h(x;x-\beta)dx - \int_\beta^{\beta+t} (\beta+t-x)\int_0^{x-\beta} h(u+\beta;u) \cdot \\ &\quad \cdot\ \frac{1}{t-u}h(x-\beta-u;t-u)dudx. \end{aligned} \tag{33}$$

Simple change of variables yields the formula

$$R_L(t;\beta) = H(\beta+t;t) - \int_0^t h(\beta+t-y;t-y)\left[e^{-\lambda y} + y\int_0^1 (1-z)h(yz;y)dz\right]dy. \tag{34}$$

Let

$$M(y) = e^{-\lambda y} + y\int_0^1 (1-z)h(yz;y)dz. \tag{35}$$

We can write then

$$R_L(t;\beta) = H(\beta+t;t) - \int_0^t M(y)h(\beta+t-y;t-y)dy. \tag{36}$$

The corresponding hazard function is

$$\begin{aligned} \Lambda(t;\beta) &= \Big\{D_0(t;\beta+t) - h(\beta+t;t) \\ &\quad + \int_0^t M(y)\Big(\frac{\partial}{\partial t}h(\beta+t-y;t-y)\Big)dy\Big\} \div R(t;\beta). \end{aligned} \tag{37}$$

7 Exponential Deterioration

In the present section we focus attention on the special case where the amount of deterioration in each shock has an exponential distribution, i.e., $F(x) = 1 - e^{-\mu(x)}$, $x \ge 0$. In this case

$$h(x;t) = \mu\sum_{n=1}^{\infty} p(n;\lambda t)\cdot p(n-1;\mu x), \tag{38}$$

and

$$\begin{aligned} H(x;t) &= e^{-\lambda t} + \sum_{n=1}^{\infty} p(n;\lambda t)(1 - P(n-1;\mu x)) \\ &= 1 - \sum_{n=1}^{\infty} p(n;\lambda t) P(n-1;\mu x) \\ &= \sum_{j=0}^{\infty} p(j;\mu x) P(j;\lambda t). \end{aligned} \tag{39}$$

7.1 Case I with $K = 1$

In the case of $K = 1$ the threshold function is

$$B(t;\tau_1,\beta_0,\beta_1) = \beta_0 I(t \le \tau) + \beta_1 I(t > \tau_1). \tag{40}$$

Let $\zeta_0 = \mu\beta_0$ and $\zeta_1 = \mu\beta_1$. The reliability function is then

$$\begin{aligned} R_1(t;\tau_1,\beta_0,\beta_1) = I(t \le \tau_1) &\sum_{n=0}^{\infty} p(n;\zeta_0) P(n;\lambda t) \\ + I(t > \tau_1) &\left[e^{-\lambda\tau_1 - \zeta_1} + \sum_{l=1}^{\infty} p(l;\zeta_1) \left(e^{-\lambda\tau_1} P(l;\lambda(t-\tau_1)) \right.\right. \\ + &\left.\left. \sum_{n=1}^{l} p(n;\lambda\tau_1) P(l-n;\lambda(t-\tau_1)) I_{\beta_0/\beta_1}(n, l-n+1) \right) \right], \end{aligned} \tag{41}$$

where $I_x(\nu_1,\nu_2)$ denotes the incomplete beta function ratio at x, $0 < x < 1$. This is the c.d.f. of beta(ν_1,ν_2).

In Figure 1 we show such a reliability function for the case of $\tau_1 = 15$, $\beta_0 = 10$, $\beta_1 = 15$, $\mu = 2$ and $\lambda = 1$.

The hazard function is

$$\Lambda_1(t;\tau_1,\beta_0,\beta_1) = \frac{-\frac{d}{dt} R_1(t;\tau_1,\beta_0,\beta_1)}{R_1(t;\tau_1,\beta_0,\beta_1)}. \tag{42}$$

Notice that $-\dfrac{d}{dt} P(n;\lambda t) = \lambda p(n;\lambda t)$. Thus, according to (41),

$$\begin{aligned} -\frac{d}{dt} R_1(t;\tau_1,\beta_0,\beta_1) = I(t \le \tau_1) \lambda &\sum_{n=0}^{\infty} p(n;\zeta_0) \cdot \\ \cdot\ p(n;\lambda t) + I(t > \tau_1) &\left[\lambda \sum_{l=1}^{\infty} p(l;\zeta_1) (e^{-\lambda\tau_1} p(l;\lambda(t-\tau_1)) \right. \\ + &\left. \sum_{n=1}^{l} p(n;\lambda\tau_1) p(l-n;\lambda(t-\tau_1)) I_{\beta_0/\beta_1}(n, l-n+1)) \right]. \end{aligned} \tag{43}$$

The hazard function is then the ratio of (43) over (41). In Figure 2 we present the hazard function corresponding to the reliability function of Figure 1.

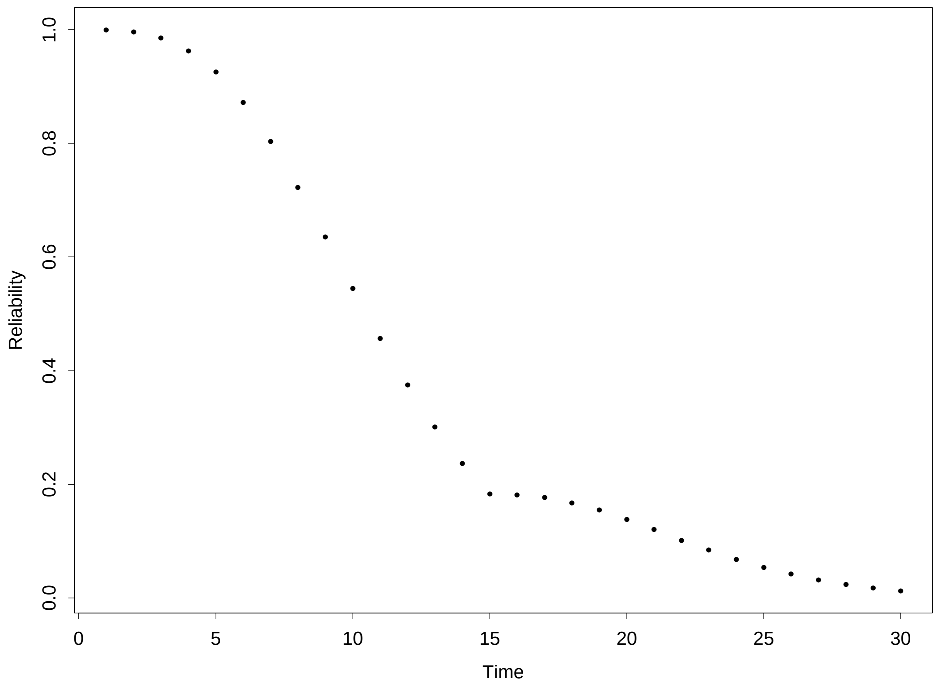

Figure 1. The Reliability Function $R_1(t; 15, 10, 15)$

7.2 Case I with $K = 2$

In the present case,

$$\begin{aligned} R_2(t; \tau_1, \tau_2, \beta_0, \beta_1, \beta_2) &= I(t \le \tau_2) R_1(t; \tau_1, \beta_0, \beta_1) \\ &+ I(t > \tau_2) \Big[e^{-\lambda \tau_2} H(\beta_2; t) \\ &+ \int_0^{\beta_1} g_2(x; \tau_1, \tau_2, \beta_0)) H(\beta_2 - x; t) dx \Big], \end{aligned} \tag{44}$$

where

$$g_2(x; \tau_1, \tau_2, \beta_0) = \int_0^{x \wedge \beta_0} h(y; \tau_1) h(x - y; \tau_2 - \tau_1) dy. \tag{45}$$

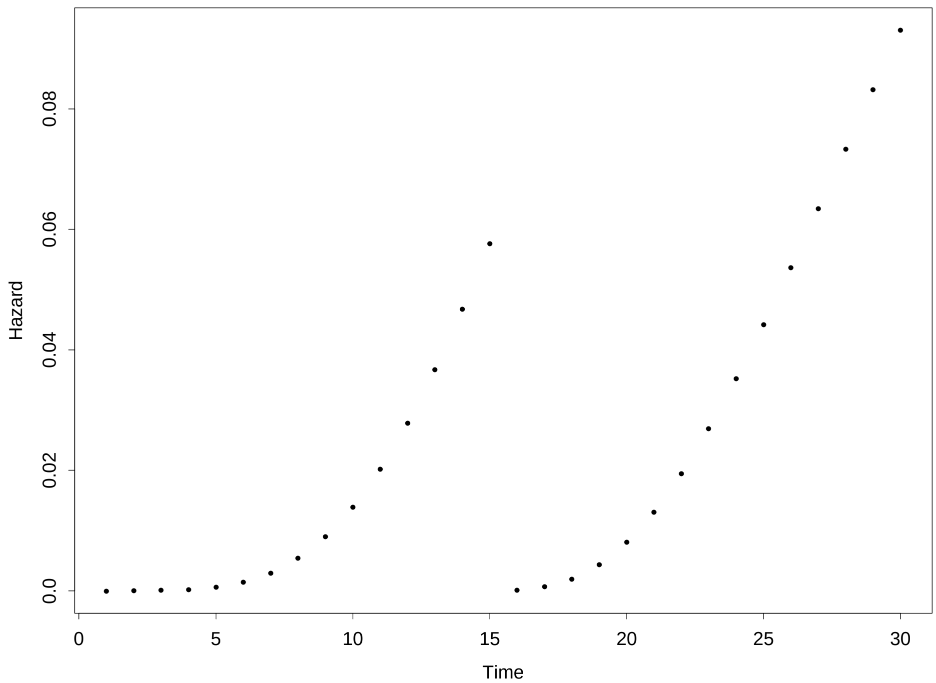

Figure 2. The Hazard Function $\Lambda_1(t; 15, 10, 15)$ with $\mu = 2$ and $\lambda = 1$

Notice that if $x \leq \beta_0$ then $g_2(x; \tau_1) = h(x; \tau_1)$. For $\beta_0 < x \leq \beta_1$ we get

$$\begin{aligned} g_2(x; \tau_1, \tau_2, \beta_0) &= \int_0^{\beta_0} h(y; \tau_1) h(x - y; \tau_2 - \tau_1) dy \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} p(n; \lambda\tau_1) p(m; \lambda(\tau_2 - \tau_1) \int_0^{\beta_0} f^{(n)}(y) f^{(m)}(x - y) dy \\ &= \mu \sum_{l=2}^{\infty} p(l-1; \mu x) e^{-\lambda\tau_2} \frac{\lambda^l}{l!} \sum_{n=1}^{l-1} \binom{l}{n} \tau_1^n (\tau_2 - \tau_1)^{l-n} I_{\beta_0/x}(n, l - n). \end{aligned} \tag{46}$$

In Figure 3 we show the reliability function $R_2(t; \tau_1, \tau_2, \beta_0, \beta_1, \beta_2)$ with $\lambda = 3$, $\mu = 2$, $\tau_1 = 10$, $\tau_2 = 15$, $\beta_0 = 10$, $\beta_1 = 15$, $\beta_2 = 20$.

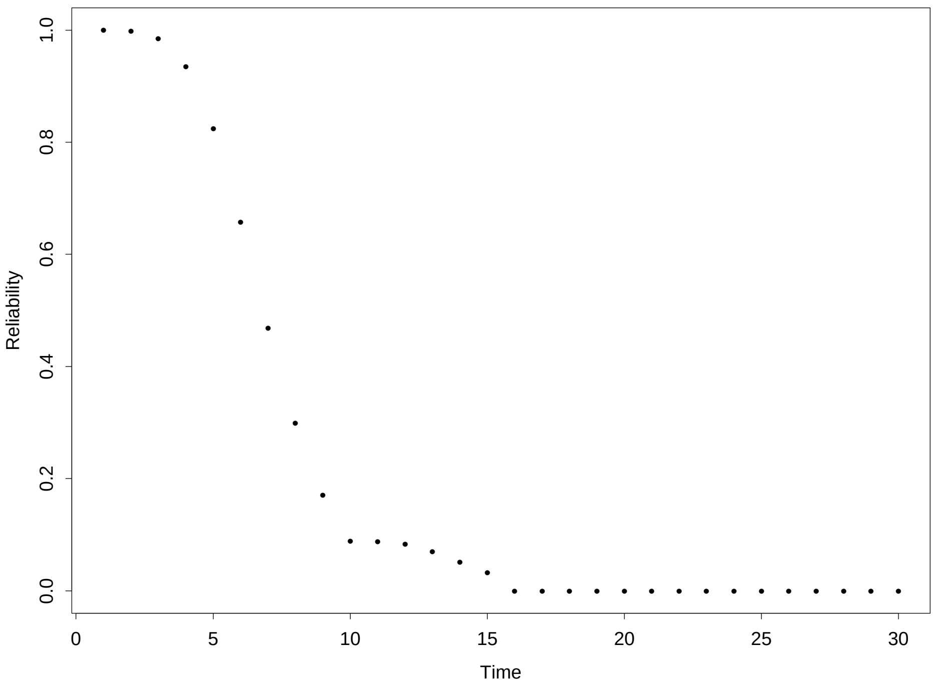

Figure 3. Reliability Function $R_2(t;\tau_1,\tau_2,\boldsymbol{\beta})$ with $K=2$

7.3 Case II with $K=1$

We assume here that the time of change τ_1 has an exponential distribution, i.e., $P(U > t) = e^{-kt}$ and $\psi(U) = ke^{-ku}$, $u > 0$. According to (17) the reliability function is

$$\begin{aligned} R_1^*(t;k,\beta_0,\beta_1) &= e^{-kt}\sum_{n=0}^{\infty} p(n;\zeta_0)P(n;\lambda t) \\ &+ k\int_0^t e^{-(k+\lambda)u}\sum_{n=0}^{\infty} p(n;\zeta_1)P(n;\lambda(t-u))du \\ &= k\int_0^t e^{-ku}\int_0^{\beta_1}\mu\sum_{n=1}^{\infty} p(n;\lambda u)p(n-1;\mu y)\cdot \\ &\cdot \sum_{m=0}^{\infty} p(m;\mu(\beta_1-y))P(m;\lambda(t-u))dydu. \end{aligned} \tag{47}$$

Define the function

$$M^*(i,j;\alpha) = \alpha\int_0^1 e^{-\alpha z}z^{i-1}(1-z)^{j-1}dz. \tag{48}$$

With this function we obtain, after some manipulations,

$$\begin{aligned} R_1^*(t;k,\beta_0,\beta_1) = e^{-kt} & \sum_{n=0}^{\infty} p(n;\zeta_0)P(n;\lambda t) \\ & + \sum_{n=0}^{\infty} p(n;\lambda t)(1-P(n-1;\zeta_1))M^*(1,n+1);kt) \\ & + \sum_{n=1}^{\infty} p(n;\zeta_1) \sum_{m=0}^{n-1}\sum_{j=0}^{m} p(n+j;\lambda t)M^*(n-m+1,j+1,kt). \end{aligned} \tag{49}$$

In Figure 4 we present the reliability function $R_1^*(t;k,\beta_0,\beta_1)$ for $k=0.1$ (points) and $k=0.5$ (line), $\lambda=\mu=2$, $\beta_0=10$, $\beta_1=15$.

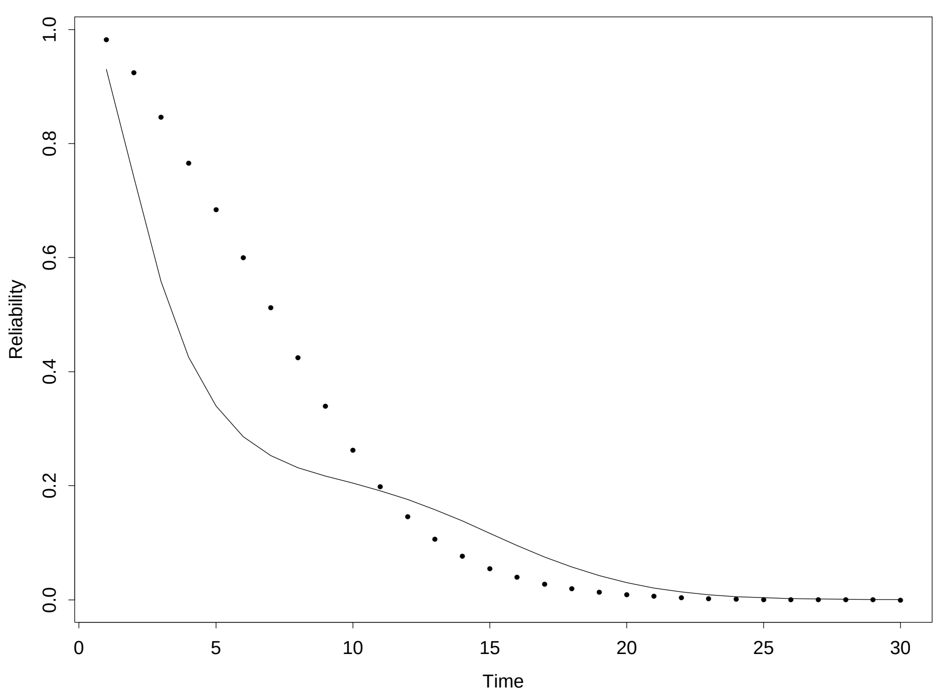

Figure 4. The Reliability function $R_1^*(t;k,\beta_0,\beta_1)$

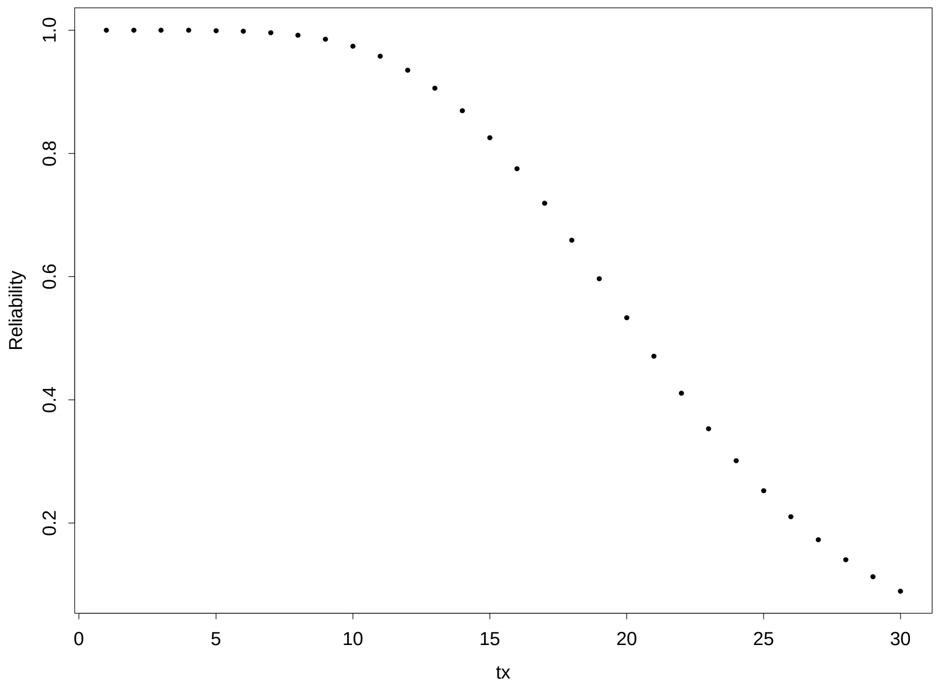

Figure 5. The Reliability Function $R_c(t;\gamma_0,\beta_0,\beta_1)$

7.4 Case III, $K=1$

When $X \sim \exp(\mu)$, the distribution of $R(T_c)$ is $\exp(\mu)$, independent of T_c. Thus, the joint p.d.f. of $(T_c, R(T_c))$ is

$$p(s,r;\gamma_0) = \psi(s;\gamma_0)\mu e^{-\mu r}, \quad 0<s<\infty, \;\; 0<r<\infty, \tag{50}$$

where the p.d.f. of T_c is

$$\psi(s;\gamma_0) = \lambda \sum_{n=0}^{\infty} p(n;\mu\gamma_0)p(n;\lambda s). \tag{51}$$

Thus, according to (27) the reliability function is

$$\begin{aligned} R_c(t;\gamma_0,\beta_0,\beta_1) = & \sum_{n=0}^{\infty} p(n;\mu\gamma_0)P(n;\lambda t) \\ & + \int_0^t \psi(s;\gamma_0)\int_0^{\beta_0-\gamma_0} \mu e^{-\mu r}\sum_{n=0}^{\infty} p(n;\mu\beta_1-\mu\gamma_0-\mu r)\,\cdot \\ & \cdot\; P(n;\lambda(t-s))drds. \end{aligned} \tag{52}$$

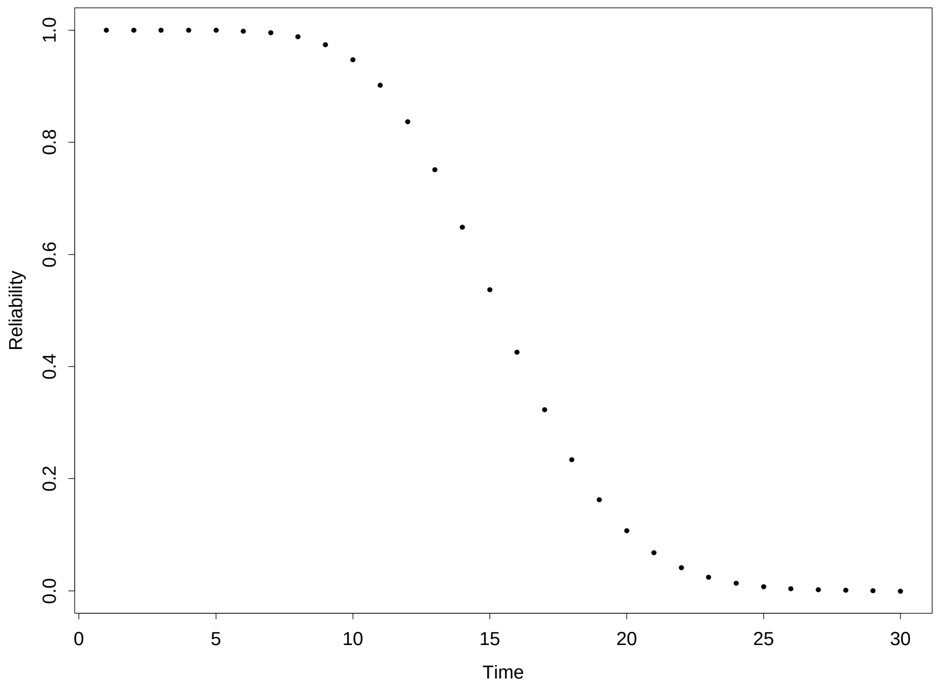

Figure 6. The Reliability Function $R_L(t;\beta)$

Explicitly we get

$$R_c(t;\gamma_0,\beta_0,\beta_1) = \sum_{n=0}^{\infty} p(n;\mu\gamma_0)P(n;\lambda t) + \lambda t \sum_{l=1}^{\infty}\sum_{m=0}^{l-1} p(l-m;\mu(\beta_0-\gamma_0)) \left(\frac{\beta_0-\gamma_0}{\beta_1-\gamma_1}\right)^{l-m} \cdot \cdot \; p(m;\mu\gamma_0)\sum_{j=0}^{l-m} p(m+j;\lambda t)/(m+j+1). \tag{53}$$

In Figure 5 we present $R_c(t;\gamma_0,\beta_0,\beta_1)$ for the values of $\lambda = 1$, $\mu = 2$, $\gamma_0 = 10$, $\beta_0 = 15$, $\beta_1 = 20$.

7.5 Case IV: Linear Threshold

According to (36)

$$
\begin{aligned}
R_L(t;\beta) = &\sum_{n=0}^{\infty} p(n;\mu(\beta+t))P(n;\lambda t) \\
&- \sum_{n=1}^{\infty} p(n;\lambda t)p(n-1;\mu(\beta+t))\mu^{n+1}t \cdot \\
&\cdot \int_0^1 \left(1 - \frac{t}{t+\beta}z\right)^{n-1} (1-z)^n e^{(\lambda+\mu)tz} M(tz)dz.
\end{aligned}
$$

In Figure 6 we present this reliability function for $\lambda = 1$, $\mu = 2$, $\beta = 10$.

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