STOCHASTIC AND DETERMINISTIC CHARACTERISTICS OF RECURSIVELY DEFINED NETWORKS

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Abstract

The calculation of stochastic and deterministic characteristics of networks is of great interest in both reliability theory and the theory of transportation networks . Some of them have exponential complexity. These problems are important in applied probability and informatics. In this paper we construct some recursive definitions, calculate different characteristics by recursive formulas and derive linear bounds for the numbers of arithmetic operations for our algorithms. The results are applied to calculations of reliability and to the solution of a salesman problem.

Keywords: Recursive definitions, reliability, shortest path, minimal weight.

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Introduction

The calculation of stochastic and deterministic characteristics of networks is of great interest in both reliability theory Barlow, Proschan (1965), Ushakov et al. (1985), Riabinin (2007) and the theory of transportation networks Belov, Vorobiev et al. (1976), Kormen, Leizerson et al. (2004) Some calculations are exceptionally complex, requiring large numbers of arithmetic operations which increase geometrically with the numbers of network arcs. These problems are important in applied probability and informatics.

But for some classes of networks, there are sufficiently fast algorithms for solving these problems. For example, simple algorithms exist to calculate reliabilities for parallel-sequential networks, based on recursive formulas. These algorithms require a linear number of arithmetic operations dependent on the numbers of network arcs. Such an approach to fast algorithms construction for recursively defined networks has still not received adequate attention.

Recursive definitions are also fundamental in different applications in informatics and have recently been used in nanotechnology. We are thus led to the idea of constructing recursive definitions, calculating different characteristics by recursive formulas and deriving linear bounds for the calculated number of arithmetic operations of our algorithms. In this paper we attempt such an approach for calculations of reliability and the solution of a salesman problem. This suggested approach may also lead to recursive formulas for the calculation of other characteristics of networks connected with physical applications, rather than with reliability theory or the theory of transportation networks.

1 Definitions

Suppose that $\Gamma = \{U, W\}$ is a nonoriented graph with a finite set of nodes U, a finite set of arcs W and dedicated initial and final nodes $u, v \in U$. Each arc $w \in W$ is characterized by a set of positive numbers: a probability of work $p_w,\ 0 < p_w < 1$, a length d_w and a weight (ability to handle) s_w. Let us introduce the following characteristics of the graph Γ:
1) its number of nodes $l(\Gamma)$ and number of arcs $m(\Gamma)$;
2) a probability $P_\Gamma = P_\Gamma(p_w,\ w \in W)$ that there is a path from u to v, and the number of arithmetic operations $n(P_\Gamma)$ necessary to calculate P_Γ;
3) a probability $P'_\Gamma = P'_\Gamma(p_w,\ w \in W)$ that there is a path from u to v through all nodes of the graph, and the number $n(P'_\Gamma)$ of arithmetic operations necessary to calculate P'_Γ;
4) a probability $P''_\Gamma = P''_\Gamma(p_w,\ w \in W)$ of a closed path through all nodes of the graph Γ, and the number

$n(P''_\Gamma)$ of arithmetic operations necessary to calculate P''_Γ;
5) the length of a shortest path $D_\Gamma = D_\Gamma(d_w,\ w \in W)$ from u to v, and the number $n(D_\Gamma)$ of arithmetic operations necessary to calculate D_Γ;
6) the length of a shortest path $D'_\Gamma = D'_\Gamma(d_w,\ w \in W)$ from u to v through all nodes of the graph Γ, and the number $n(D'_\Gamma)$ of arithmetic operations necessary to calculate D'_Γ;
7) the length $D''_\Gamma = D''_\Gamma(d_w,\ w \in W)$ of a shortest closed path through all nodes of the graph Γ, and the number $n(D''_\Gamma)$ of arithmetic operations necessary to calculate D''_Γ;
8) the minimal weight for cross sections $S_\Gamma = S_\Gamma(s_w,\ w \in W)$ from u to v, and the number $n(S_\Gamma)$ of arithmetic operations necessary to calculate S_Γ.
If in the graph Γ, there is not a path connecting u, v (through all graph nodes) then

$$P_\Gamma = 0,\ D_\Gamma = \infty,\ S_\Gamma = 0\ (P'_\Gamma = 0,\ D'_\Gamma = \infty).$$

2 Ports constructed by a replacement of arcs

Suppose that $\mathcal{B}_*$ is a family of ports Γ with nonintersecting sets of nodes. Define a class $\mathcal{B}$ of ports with a set of generators $\mathcal{B}_*,\ \mathcal{B}_* \subset \mathcal{B}$ by the following condition. If a port $\Gamma = \{U, W\} \in \mathcal{B}_*$ with $W = \{w_1, \dots, w_m\}$, and ports $\Gamma_1 = \{U_1, W_1\}, \dots, \Gamma_m = \{U_m, W_m\} \in \mathcal{B}$ with $U_1 \cap \ldots \cap U_m = \emptyset$, then a port $\Gamma' = \Gamma(\Gamma_1, \dots, \Gamma_m)$ constructed from Γ by replacing arcs $w_1, \dots, w_m$ by ports $\Gamma_1, \dots, \Gamma_m$, also belongs to the class $\mathcal{B}$.

Algorithms for calculating the reliability P, the length of a shortest path D, the minimal weight S for ports from the class $\mathcal{B}$ are based on the recursive formulas

$$P_{\Gamma'} = P_\Gamma(P_{\Gamma_1}, \dots, P_{\Gamma_m}),\ D_{\Gamma'} = D_\Gamma(D_{\Gamma_1}, ..., D_{\Gamma_m}),\ \ S_{\Gamma'} = S_\Gamma(S_{\Gamma_1}, ..., S_{\Gamma_m}). \tag{1}$$

A calculation of the complexity of these algorithms is defined by the following statement.

Theorem 1. *Suppose that* $\inf\limits_{\Gamma \in \mathcal{B}_*} m(\Gamma) > 1$. *For any* $\Gamma \in \mathcal{B}$ *we have:*
1) if arcs of Γ *work independently and* $\sup\limits_{\Gamma \in \mathcal{B}_*} n(P_\Gamma) < \infty,\ \sup\limits_{\Gamma \in \mathcal{B}_*} n(P'_\Gamma) < \infty$ *then*

$$n(P_\Gamma) \le (m(\Gamma) - 1) \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma),\ n(P'_\Gamma) \le (m(\Gamma) - 1) \sup_{\Gamma \in \mathcal{B}_*} n(P'_\Gamma); \tag{2}$$

2) if $\sup\limits_{\Gamma \in \mathcal{B}_*} n(D_\Gamma) < \infty,\ \sup\limits_{\Gamma \in \mathcal{B}_*} n(D'_\Gamma) < \infty$ *then*

$$n(D_\Gamma) \le (m(\Gamma) - 1) \sup_{\Gamma \in \mathcal{B}_*} n(D_\Gamma),\ n(D'_\Gamma) \le (m(\Gamma) - 1) \sup_{\Gamma \in \mathcal{B}_*} n(D'_\Gamma); \tag{3}$$

3) if $\sup\limits_{\Gamma \in \mathcal{B}_*} n(S_\Gamma) < \infty$ *then*

$$n(S_\Gamma) \le (m(\Gamma) - 1) \sup_{\Gamma \in \mathcal{B}_*} n(S_\Gamma). \tag{4}$$

Proof. We prove the inequality

$$n(P_\Gamma) \le (m(\Gamma) - 1) \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma),$$

from which all other statements of this theorem may be established analogously. From the conditions of the theorem for $\Gamma \in \mathcal{B}_*$ the inequality (2) is true. Suppose that the inequality (2) is true for $\Gamma_1, \dots, \Gamma_m \in \mathcal{B}$ and $\Gamma' = \Gamma(\Gamma_1, \dots, \Gamma_m)$. Then from the equations (1) and the equality $m(\Gamma') = m(\Gamma_1) + \ldots + m(\Gamma_m)$, we obtain that

$$n(P_{\Gamma'}) = n(P_{\Gamma_1}) + \ldots + n(P_{\Gamma_m}) + n(P_\Gamma),$$

$$n(P_{\Gamma'}) \le \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma)(m(\Gamma_1) - 1 + \ldots + m(\Gamma_m) - 1 + 1) \le (m(\Gamma') - 1) \sup_{\Gamma \in \mathcal{B}_*} n(P_\Gamma).$$

□

From the inequalities (2) we see that to calculate the probabilities $P_\Gamma,\ P'_\Gamma$, linear numbers of arithmetic operations $n(P_\Gamma),\ n(P'_\Gamma)$ are necessary. Note that for ports of general type, $n(D_\Gamma)$ increases as a square of $m(\Gamma)$, while $n(S_\Gamma)$ increases as a cube of $m(\Gamma)$, $n(P_\Gamma),\ n(P'_\Gamma)$ and $n(D'_\Gamma)$ increase as geometric progressions of $m(\Gamma)$ Barlow, Proschan (1965), Ushakov et al. (1985), Kormen, Leizerson et al. (2004).

3 Networks constructed by clustering of nodes

A salesman problem. Suppose that $\mathcal{D}_*$ is a family of networks Γ with nonintersecting sets of arcs. Define recursively a class of networks $\mathcal{D},\ \mathcal{D}_* \subset \mathcal{D}$ by the following condition. If for a pair of networks $\Gamma_1 = \{U_1, W_1\} \in \mathcal{D},\ \Gamma_2 = \{U_2, W_2\} \in \mathcal{D}_*,\ W_1 \cap W_2 = \emptyset,\ U_1 \cap U_2 = \{z\}$ (with a single node z) then $\Gamma_1 \cup \Gamma_2 \in \mathcal{D}$.

Define recursively a number $k(\Gamma),\ \Gamma \in \mathcal{D}$:

$$k(\Gamma) = \begin{cases} 1,\ \Gamma \in \mathcal{D}_*, \\ k(\Gamma_1) + k(\Gamma_2),\ \Gamma = \Gamma_1 \cup \Gamma_2,\ \Gamma_1 \in \mathcal{D},\ \Gamma_2 \in \mathcal{D}_*. \end{cases}$$

It is clear that $k(\Gamma) \leq l(\Gamma)$.

Algorithms of the reliability P'' and calculations of the length of a shortest path D'' for networks from the class $\mathcal{D}$ are based on the recursive formulas

$$P''_{\Gamma_1 \cup \Gamma_2} = P''_{\Gamma_1} P''_{\Gamma_2},\ D''_{\Gamma_1 \cup \Gamma_2} = D''_{\Gamma_1} + D''_{\Gamma_2}. \tag{5}$$

A calculation of the complexity of these algorithms is defined by the following statement.

Theorem 2. *Suppose that $\Gamma_1, ..., \Gamma_l$ is a final family of networks with nonintersecting sets of arcs. If $\mathcal{D}_*$ consists of $\Gamma_1, ..., \Gamma_l$ independent copies, then for any $\Gamma \in \mathcal{D}$:*

$$n(P''_\Gamma) \leq k(\Gamma) + \sum_{i=1}^{l} n(P''_{\Gamma_i}),\ n(D''_\Gamma) \leq k(\Gamma) + \sum_{i=1}^{l} n(D''_{\Gamma_i}). \tag{6}$$

Proof. We prove the first inequality in (6), the second inequality may be proved analogously. It is clear that this inequality is true for all $\Gamma \in \mathcal{D}_*$.

Suppose that $\Gamma_1 \in \mathcal{D},\ \Gamma_2 \in \mathcal{D}_*,\ R$ is a closed path through all nodes of the network $\Gamma_1 \cup \Gamma_2$ and z is its initial node. Divide R into closed paths with the initial node z, which belong entirely to Γ_1 or to Γ_2. Connect all closed paths which belong to Γ_1 and construct a closed path which passes through all nodes of Γ_1. Analogously construct a closed path which passes through all nodes of Γ_2. Then $P''_{\Gamma_1 \cup \Gamma_2} = P''_{\Gamma_1} P''_{\Gamma_2}$.

Suppose that the first inequality from (6) is true for Γ_1, then from $P''_{\Gamma_1 \cup \Gamma_2} = P''_{\Gamma_1} P''_{\Gamma_2}$

$$n(P''_{\Gamma_1 \cup \Gamma_2}) \leq \sum_{i=1}^{l} n(P''_{\Gamma_i}) + k(\Gamma_1) + 1 = \sum_{i=1}^{l} n(P''_{\Gamma_i}) + k(\Gamma_1 \cup \Gamma_2).$$

□

This analog of the salesman problem has a linear solution complexity depending on the number of nodes $l(\Gamma)$.

A problem of Floid and Steinberg. In the problem considered in Floid, Steinberg (1975) complete families of

$$\{D_\Gamma,\ u, v \in U,\ u \neq v\},\ \{S_\Gamma,\ u, v \in U,\ u \neq v\}$$

but not their elements are calculated. In this paper families

$$\{D_\Gamma,\ u, v \in U,\ u \neq v\},\ \{S_\Gamma,\ u, v \in U,\ u \neq v\},\ \{P_\Gamma,\ u, v \in U,\ u \neq v\}$$

are calculated on the basis of recursive formulas: suppose that $\Gamma' \in \mathcal{D},\ \Gamma'' \in \mathcal{D}_*,\ U' \cap U'' = \{z\}$, then

$$D_{\Gamma'\cup\Gamma''} = \begin{cases} D_{\Gamma'},\ u,v \in U', \\ D_{\Gamma''},\ u,v \in U'', \\ D_{\Gamma'} + D_{\Gamma''},\ u \in U',\ v \in U'', \end{cases} \qquad S_{\Gamma'\cup\Gamma''} = \begin{cases} S_{\Gamma'},\ u,v \in U', \\ S_{\Gamma''},\ u,v \in U'', \\ \min(S_{\Gamma'}, S_{\Gamma''}),\ u \in U',\ v \in U'', \end{cases} \tag{7}$$

$$P_{\Gamma'\cup\Gamma''} = \begin{cases} P_{\Gamma'},\ u,v \in U', \\ P_{\Gamma''},\ u,v \in U'', \\ P_{\Gamma'}P_{\Gamma''},\ u \in U',\ v \in U''. \end{cases}$$

In the last equalities which recursively define D, S, P, the quantities $D_{\Gamma'},\ S_{\Gamma'},\ P_{\Gamma'}$ characterize connections between nodes u, z, while the quantities $D_{\Gamma''},\ S_{\Gamma''},\ P_{\Gamma''}$ characterize connections between nodes z, v. A calculation of the complexity of algorithms based on recursive formulas (7) is defined by the following statement.

Theorem 3. *Under the conditions of theorem 2 for any* $\Gamma \in \mathcal{D}$:

$$\frac{l(\Gamma)(l(\Gamma)-1)}{2} \leq \sum_{u,v\in U,\ u\neq v} n(D_\Gamma) \leq \frac{l(\Gamma)(l(\Gamma)-1)}{2} + \sum_{i=1}^{l} \sum_{u,v\in U_i,\ u\neq v} n(D_{\Gamma_i}). \tag{8}$$

$$\frac{l(\Gamma)(l(\Gamma)-1)}{2} \leq \sum_{u,v\in U,\ u\neq v} n(S_\Gamma) \leq \frac{l(\Gamma)(l(\Gamma)-1)}{2} + \sum_{i=1}^{l} \sum_{u,v\in U_i,\ u\neq v} n(S_{\Gamma_i}). \tag{9}$$

$$\frac{l(\Gamma)(l(\Gamma)-1)}{2} \leq \sum_{u,v\in U,\ u\neq v} n(P_\Gamma) \leq \frac{l(\Gamma)(l(\Gamma)-1)}{2} + \sum_{i=1}^{l} \sum_{u,v\in U_i,\ u\neq v} n(P_{\Gamma_i}). \tag{10}$$

Proof. Suppose that the inequality (8) is true for Γ', then from the recursive formulas (7) and from the equality $l(\Gamma' \cup \Gamma'') = l(\Gamma') + l(\Gamma'') - 1$ we obtain

$$\sum_{u,v\in U'\cup U'',\ u\neq v} n(D_{\Gamma'\cup\Gamma''}) \leq \sum_{i=1}^{l} \sum_{u,v\in U_i,\ u\neq v} n(D_{\Gamma_i}) + \frac{l(\Gamma_1)(l(\Gamma_1)-1)}{2} + \frac{l(\Gamma_2)(l(\Gamma_2)-1)}{2} +$$

$$+(l(\Gamma_1 - 1))(l(\Gamma_2) - 1) = \sum_{i=1}^{l} \sum_{u,v\in U_i,\ u\neq v} n(D_{\Gamma_i}) + \frac{l(\Gamma_1 \cup \Gamma_2)(l(\Gamma_1 \cup \Gamma_2) - 1)}{2}.$$

Analogous inequalities may be obtained for S, P. □

From the inequalities (8)-(10) we obtain:

$$\lim_{l(\Gamma)\to\infty} \frac{\sum_{u,v\in U,\ u\neq v} n(D_\Gamma)}{\frac{l(\Gamma)(l(\Gamma)-1))}{2}} = \lim_{l(\Gamma)\to\infty} \frac{\sum_{u,v\in U,\ u\neq v} n(S_\Gamma)}{\frac{l(\Gamma)(l(\Gamma)-1))}{2}} = \lim_{l(\Gamma)\to\infty} \frac{\sum_{u,v\in U,\ u\neq v} n(P_\Gamma)}{\frac{l(\Gamma)(l(\Gamma)-1))}{2}} = 1. \tag{11}$$

Hence, asymptotically for $l(\Gamma) \to \infty$, to calculate the length of the shortest path D or a minimal weight S, or a reliability P for a single pair of initial and final nodes, only a single arithmetic operation is necessary.

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