Recursive Methods for Fuzzy Multi-State k-out-of-n:G System Reliability Assessment

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Abstract

Compared with a binary system model, a multi-state system model provides a more flexible tool for representing engineering systems in real life. In the classic multi-state system theory, it is assumed that exact probability and performance level of each component state are given. However it may be difficult to obtain sufficient data to estimate the precise values of these probabilities and performance levels in many engineering systems. In some recent research fuzzy theories have been used in the multi-state system models to overcome these deficiencies. The definition of fuzzy multi-state k-out-of-n:G system is proposed in this paper. Recursive algorithms are provided for evaluating reliabilities of the proposed system.

1 Introduction

The k-out-of-n system model finds wide applications in both industry and military [1] and has been extensively studied for many years. A binary k-out-of-n:G system works if and only if at least k components work. The natural extensions of binary k-out-of-n systems are the models of multi-state k-out-of-n systems, which have been developed in [2] by allowing for different k values with respect to different states. Recursive methods are the primary algorithms for exact performance evaluation of multi-state k-out-of-n system.

In the classic multi-state theory, it is assumed that exact state probability and performance level of each component of multi-state k-out-of-n systems are given. However, in many cases it is difficult to obtain sufficient data to estimate the precise values of these probabilities and performance levels in these engineering systems. Moreover, inaccuracy in system models that is caused by human errors is difficult to deal with solely by means of the conventional reliability theory [3].

There are few studies focusing on reliability assessment of multi-state systems (MSS) by using fuzzy set theory. The authors of [4] have made an attempt at this problem. In [4], the basic definition of fuzzy multi-state system (FMSS) model is also given: the state probability and/or the state performance level (rate) of a system component can be represented as fuzzy values. In this paper, the definition of multi-state k-out-of-n:G system is extended from the crisp domain to the fuzzy domain and the model of fuzzy multi-state k-out-of-n:G system is proposed. The recursive methods are developed and used to evaluate reliabilities of such systems. In the following sections, the fuzzy multi-state k-out-of-n:G system is defined in Section 2; the corresponding performance distribution are defined and assessed by using proposed recursive algorithms in Section 3.

Notation:

ϕ	MSS structure function, which represents the system performance levels taking crisp values
i	component index
j	state index, a crisp value taking integer values only for MSS or FMSS
M	the highest possible state of each component and system if the MSS is homogenous
g_{ij}	performance level of component i in state j, which is a crisp value
p_{ij}	probability of component i in state j, which is a crisp value
G_i	performance level of component j taking crisp values, $G_i = g_{ij}$ if component i is in state j, $\quad 0 \le j \le M_i$
k_j	the minimum system performance level required to ensure the system in state j or above, which is a crisp value

$\tilde{\phi}$	FMSS structure function, which represents the system performance levels taking fuzzy values
$\tilde{g}_{ij}$	performance level of component i in state j, which is represented as a fuzzy value
$\tilde{p}_{ij}$	probability of component i in state j, which is represented as a fuzzy value
$\tilde{G}_i$	fuzzy performance level of component j, $\tilde{G}_i = \tilde{g}_{ij}$ if component i is in state j, $0 \leq j \leq M_i$
$\tilde{G}$	fuzzy n-dimensional vector, which represents fuzzy performance levels of all components
$\tilde{k}_j$	the minimum system performance level required to ensure the system in state j or above, which can be represented as a fuzzy value
$\tilde{R}_j(\tilde{k}_j, i)$	fuzzy probability for the system to be in state j or above when there are i components in the system

2 Definition of Fuzzy Multi-State k-out-of-n:G System

The k-out-of-n:G system structure is a very popular type of redundancy in many fault-tolerant systems, which finds wide application in both industry and military [1]. In the binary context, a k-out-of-n:G system with n components works if and only if at least $k$components work. Both series and parallel systems are special cases of the k-out-of-n:G system (note that $k = n$ corresponds to the binary series system and $k = 1$ corresponds to the binary parallel system) [1, 5]. The definition of k-out-of-n:G system has been extended to the multi-state context by allowing both the system and its component to have $M + 1$ states such that $M \geq 2$ [1]. For example, reference [6] defines a general multi-state k-out-of-n:G system: in a system with n components each component and the system can be in $M + 1$ states; each component, when in state j, has a performance value; the system is in state j or above if the total performance value of all components is greater than or equal to a pre-defined value of state k_j.

In this section we define the fuzzy multi-state k-out-of-n:G system. In the definition each component's performance level of every possible state contributing to the system performance level can be measured as a fuzzy value and the system being in or above a certain state level requires the system performance level is greater than or equal to a pre-defined value. In some cases determining precisely these pre-defined values for different system sates may be difficult. Therefore these values can be not only crisp numbers but also fuzzy values. The formal definition of the fuzzy multi-state k-out-of-n:G system is given below.

Definition 1. In a system with n components each component and the system can be in $M+1$ possible states. The probability ($\tilde{p}_{ij}$) and the performance level ($\tilde{g}_{ij}$) of component i ($1 \leq i \leq n$) in state j can be measured as fuzzy values. The system performance level is the total performance level of components and can be evaluated as: $\tilde{\phi}(\tilde{G}) = \sum_{i=1}^{n} \tilde{g}_{ij}$. The system is in state j or above if $\tilde{\phi}(\tilde{G})$ is greater than or equal to $\tilde{k}_j$, a predefined value which can be a fuzzy or crisp value.

3 Recursive Algorithms

Recursive algorithms and universal generating functions (UGF) are the two primary approaches for reliability evaluation of multi-state k-out-of-n systems. In some cases, the recursive algorithms have better computational efficiency than the UGF for reliability evaluation of conventional k-out-of-n:G system. The fuzzy universal generating functions (FUGF) developed in [4] can be used to evaluate the defined fuzzy multi-state k-out-of-n:G system. In this section recursive algorithms proposed in [6] for evaluating conventional k-out-of-n systems are extended for the reliability and performance evaluation of the defined system.

A conventional recursive equation for evaluation of the system state distribution is as follows [6]:

$$R_j(k_j, i) = \sum_{r=0}^{M} p_{ir} \cdot R_j(k_j - g_{ir}, i - 1), \tag{1}$$

where $R_j(k_j, i)$ is the probability for the system to be in state j or above when there are i components in the system.

At each recursion, the boundary condition will be checked. If the condition is reached, the recursion will end. If not, the recursion will continue. The boundary conditions for this recursive equation have two possible cases: (1) when $i = 0$ and $k_j > 0$, $R_j(k_j, i) = 0$; (2) when $i \geq 0$ and $k_j \leq 0$, $R_j(k_j, i) = 1$.

The recursive methods are extended from the conventional recursive methods and used to evaluate reliabilities of the defined system. A recursive equation for evaluation of the system state distribution for the defined system is as follows:

$$\tilde{R}_j(\tilde{k}_j, i) = \sum_{r=0}^{M} \tilde{p}_{ir} \cdot \tilde{R}_j(\tilde{k}_j - \tilde{g}_{ir}, i-1), \tag{2}$$

where $\tilde{R}_j(\tilde{k}_j, i)$ is the fuzzy probability for the system to be in state j or above when there are i components in the system.

The boundary conditions for this recursive equation have three possible cases:

1. when $i = 0$ and $\tilde{k}_j$ is definitely greater than 0. In this case, the minimum system performance level required to ensure the system in state j cannot be satisfied, therefore: $\tilde{R}_j(\tilde{k}_j, i) = 0$;

2. when $i \geq 0$ and $\tilde{k}_j$ is definitely less than or equal to 0. In this case, the minimum system performance level required to ensure the system in state j can be satisfied, therefore: $\tilde{R}_j(\tilde{k}_j, i) = 1$;

3. when $i = 0$ and $\tilde{k}_j$ has the possibility of being less than or equal to 0. In this case, there exists the uncertainty whether the minimum system performance level required to ensure the system in state j can be satisfied. This uncertainty can be evaluated by possibility measures, which can be easily integrated with fuzzy sets. Therefore $\tilde{R}_j(\tilde{k}_j, i) = |\tilde{s}_j|_{rel}$; $|\tilde{s}_j|_{rel}$ is the possibility that $\tilde{k}_j$ is less than or equal to 0, such that $0 \leq |\tilde{s}_j|_{rel} \leq 1$. The calculation of $|\tilde{s}_j|_{rel}$ is summarized below.

   In order to calculate $|\tilde{s}_j|_{rel}$, the cardinality of fuzzy set [7] $\tilde{k}_j$ should be calculated first. When $i = 0$, if the membership function of $\tilde{k}_j$ is discrete, the cardinality of fuzzy set $\tilde{k}_j$ is:

$$\left|\tilde{k}_j\right| = \sum_{\tilde{k}_j \in K_j} \mu_{\tilde{k}_j}(k_j). \tag{3}$$

   When $i = 0$, if the membership function of $\tilde{k}_j$ is continuous, the cardinality of fuzzy set $\tilde{k}_j$ is:

$$\left|\tilde{k}_j\right| = \int_{k_j \in K_j} \mu_{\tilde{k}_j}(k_j) \cdot dk_j, \tag{4}$$

   where K_j is the definition domain of $\tilde{k}_j$. Define the following:

$$S_j = \{k_j \in K_j | k_j \leq 0.\}, \tag{5}$$

$$\tilde{s}_j = \{s_j, \mu(s_j) | \mu(s_j) = \mu(k_j), s_j \in S_j\}. \tag{6}$$

   If the membership function of $\tilde{k}_j$ is discrete, the cardinality of fuzzy set S_j is:

$$|\tilde{s}_j| = \sum_{s_j \in S_j} \mu_{\tilde{s}_j}(s_j). \tag{7}$$

   If the membership function of $\tilde{k}_j$ is continuous, the cardinality of fuzzy set $\tilde{s}_j$ is:

$$|\tilde{s}_j| = \int_{s_j \in S_j} \mu_{\tilde{s}_j}(s_j) \cdot ds_j. \tag{8}$$

   The relative cardinality of fuzzy set $\tilde{s}_j$ is defined as:

$$|\tilde{s}_j|_{rel} = |\tilde{s}_j| \Big/ \left|\tilde{k}_j\right|. \tag{9}$$

Suppose $\tilde{k}_j$ can be represented as a triangular fuzzy number parameterized by a triplet (b_1, b_2, b_3) as shown in Fig. 1. In this case,

$$\left|\tilde{k}_j\right| = (b_3 - b_1)/2; \qquad |\tilde{s}_j| = \frac{(b_1)^2}{2 \cdot (b_2 - b_1)}; \qquad |\tilde{s}_j|_{rel} = |\tilde{s}_j| \Big/ \left|\tilde{k}_j\right| = \frac{(b_1)^2}{(b_2 - b_1) \cdot (b_3 - b_1)}.$$

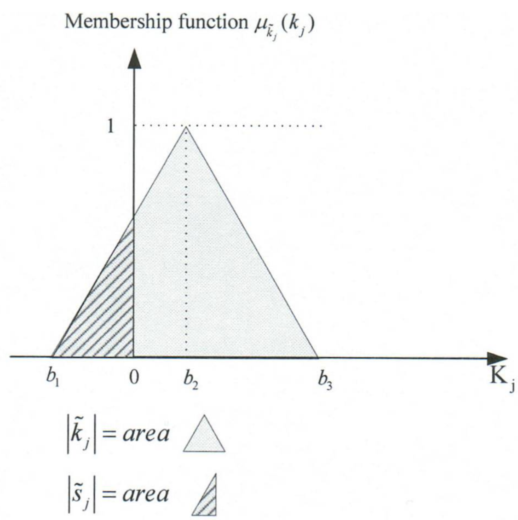

Figure 1: The minimum weight required to ensure the system in state j.

4 Conclusion

In this paper, the definition of fuzzy multi-state k-out-of-n:G system model is proposed. The recursive methods are developed and used to evaluate reliabilities of such systems.

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